The *p*-center problem with uncertainty in the demands

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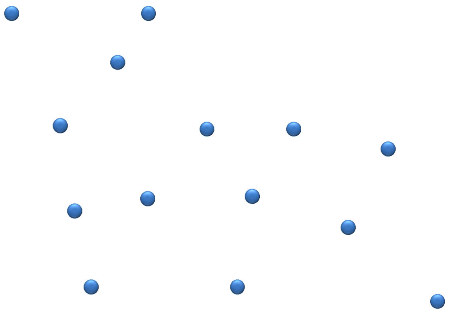
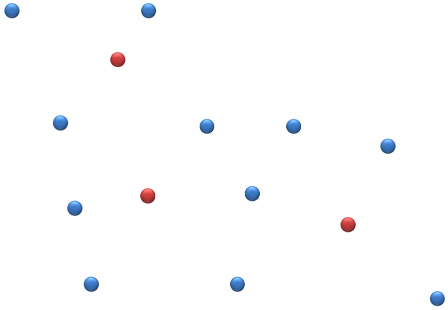
The Probabilistic p-Center Problem



The Probabilistic p-Center Problem



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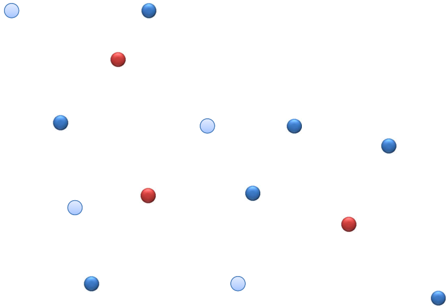

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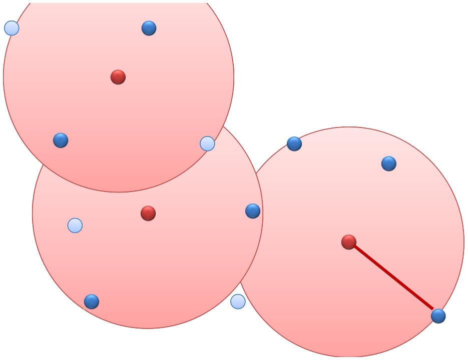

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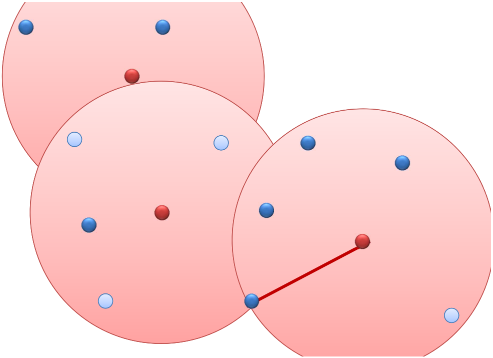

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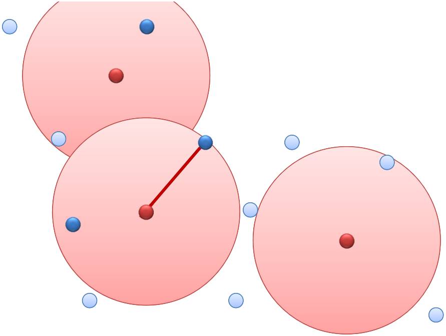

Description

- Given:
  - a set of customers I with associated demand probabilities q_j
  - a set of candidate locations J (often $I = J$)
  - a fixed number of facilities to open p
- Decide:
- After that:
- Objective:

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  - a fixed number of facilities to open p
- Decide:
  - The set of p locations
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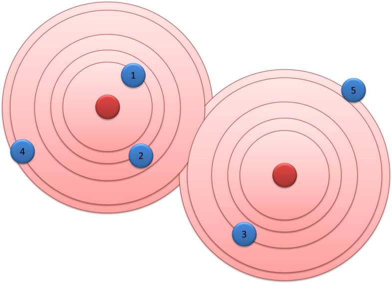
  - Only a subset of the customers will require being served Independently
- Objective:
  Minimize the expected maximum distance from a demand customer to a facility

PpCP : The objective function

Maximum service distance:

$d_{(5)}$ If 5 has demand



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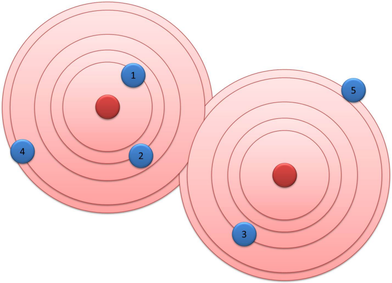
Maximum service distance:

$$d_{(5)} q_5$$



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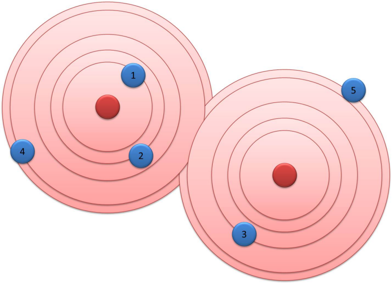


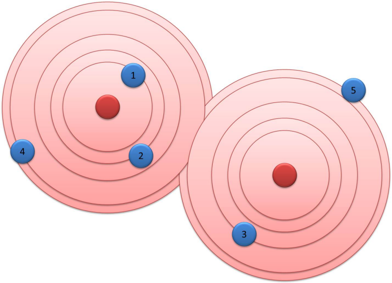
$d_{(5)} q_5$

$+$

$d_{(4)}$ If 5 hasn't and 4 does

P*p*CP : The objective function

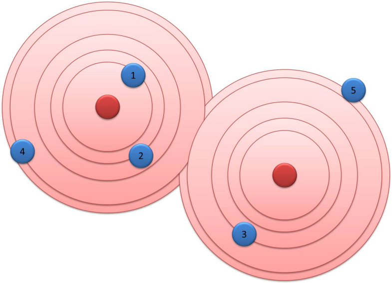
Maximum service distance:



$$d_{(5)} q_5 + d_{(4)} (1 - q_5) q_4$$

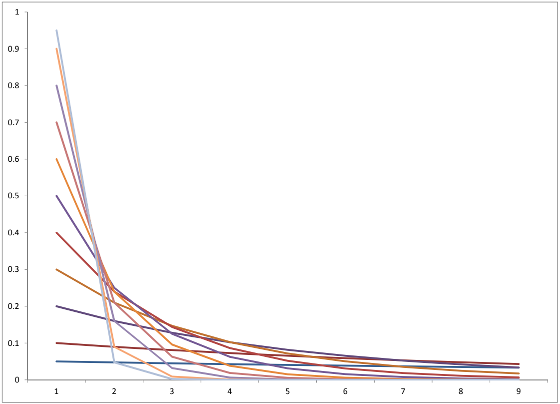
PpCP : The objective function

Maximum service distance:

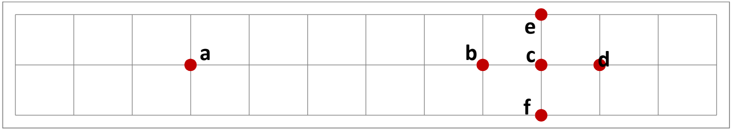


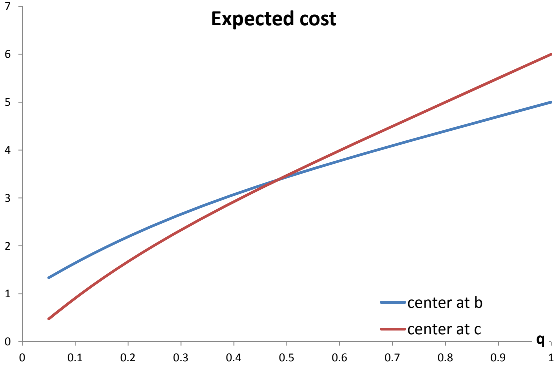
$$d_{(5)} q_5$$
$$+$$
$$d_{(4)} (1 - q_5) q_4$$
$$+$$
$$d_{(3)} (1 - q_5)(1 - q_4) q_3$$
$$+$$
$$d_{(2)} (1 - q_5)(1 - q_4)(1 - q_3) q_2$$
$$+$$
$$d_{(1)} (1 - q_5)(1 - q_4)(1 - q_3)(1 - q_2) q_1$$

Even if $q_j = q$...

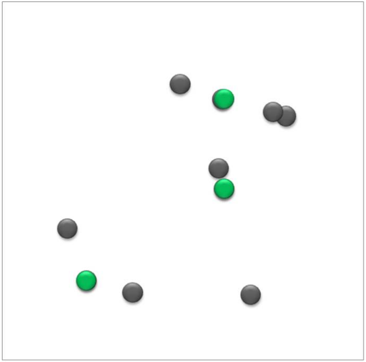


Even if $q_j = q$...

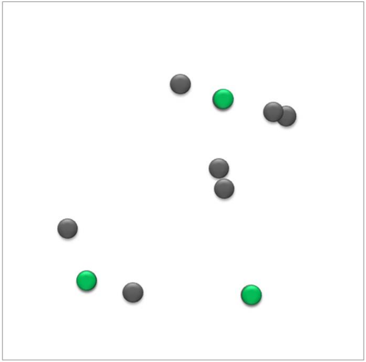

e
a
b
c
d
f


Expected cost
7
6
5
4
3
2
1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
q
center at b
center at c

Even if $q_j = q$...



3-median sol.
Optimal for q < 0.25



3-center sol.
Optimal for q ≥ 0.25

Closest assigments

Result 1: The optimal value of the PpCP is achieved in a solution where every site is covered by its closest plant.

Result 2: The optimal value of the K-PpCP need closest assigments constraints to allocate site to its closest plant.

Formulation with 4-indices variables (4F)

- $y_j = \begin{cases} 1, & \text{if a plant is opened at site } j, \\ 0, & \text{otherwise.} \end{cases}$
- π_{ij}= probability that d_{ij} is the largest service distance.
- For $i, j, k, \ell = 1, \ldots, n$ with $\{d_{ij} > d_{k\ell}\}$ or $\{d_{ij} = d_{k\ell} \text{ \& } i > k\}$:

$$x_{ijk\ell} = \begin{cases} 1, & \text{if } i \to j,\ k \to \ell \text{ and} \\ & d_{ij} \text{ right after } d_{k\ell} \text{ in sorted assignment costs} \\ 0, & \text{otherwise.} \end{cases}$$

Formulation with 4-indices variables (4F)

min $\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} d_{ij}$

s.t.

p open facilities

x variables are consistent

only assignments to open facilities

π variables are consistent with x

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min $\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} d_{ij}$

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π variables are consistent with x

$$\pi_{k\ell} \geqslant \frac{1-q_i}{q_i} q_k \pi_{ij} - 1 + x_{ijk\ell}, \qquad i,j,k,\ell = 1,\ldots,n$$

$$\pi_{ij} \geqslant q_i \left(\sum_{k=1}^{n} \sum_{\ell=1}^{n} x_{ijk\ell} - \sum_{k=1}^{n} \sum_{\ell=1}^{n} x_{k\ell ij} \right) \qquad i,j = 1,\ldots,n$$

Formulation with 4-indices variables ($4F^K$) K-PpCP

- Consider only the K largest distances (only the K furthest sites can fail).

- We add variables:

$$z_{kl} = \begin{cases} 1, & \text{if } k \text{ is allocated to } l \text{ and the distance } d_{kl} \text{ is among} \\ & \text{the } n-K \text{ smallest distances,} \\ 0, & \text{otherwise.} \end{cases}$$

Formulation with 4-indices variables ($4F^K$) K-PpCP

min $\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} d_{ij}$

s.t.

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CAC

π variables are consistent with x

Formulation with 4-indices variables ($4F^K$) K-PpCP

min $\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} d_{ij}$

s.t.

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CAC

π variables are consistent with x

$$\pi_{k\ell} \geqslant \frac{1-q_i}{q_i} q_k \pi_{ij} - 1 + x_{ijk\ell} - z_{k\ell}, \forall i,j,k,\ell \in N,$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\substack{k'=1,\ell'=1 \\ d_{k'\ell'} \geqslant d_{k\ell}}}^{n} x_{ijk'\ell'} \geq K z_{k\ell}, \qquad \forall k,\ell \in N.$$

Formulation with 3-indices variables ($3F^K$) K-PpCP

- $x_{ij}^t = \begin{cases} 1 & \text{site } i \text{ covered by plant } j \text{ and } d_{ij} \text{ is in the} \\ & \quad t\text{-th position of the (inc.) distance vector.} \\ 0 & \text{otherwise,} \end{cases}$
  $t = n - K + 1, \ldots, n.$

- $x_{ij}^{n-K} = \begin{cases} 1, & \text{if } i \text{ is allocated to } j \text{ and } d_{ij} \text{ is in a position} \\ & \quad \text{of the distance vector lower than } n - K + 1, \\ 0, & \text{otherwise,} \end{cases}$

- $\lambda_{ij}^t = x_{ij}^t \cdot \mathbb{P}(d_{ij} \text{ be the largest service cost})$

Formulation with 3-indices variables ($3F^K$) K-PpCP

min $\sum_{t=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij}^{t} d_{ij}$

s.t.

p open facilities

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CAC

λ variables self-consistent and also with x

Formulation with 3-indices variables ($3F^K$) K-PpCP

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CAC

λ variables self-consistent and also with x

$$\lambda_{ij}^{t} \leqslant x_{ij}^{t} \qquad t,i,j = 1,\ldots,n$$

$$x_{ij}^{n-K} d_{ij} \leqslant \sum_{k=1}^{n} \sum_{l=1}^{n} x_{kl}^{n-K+1} d_{kl}, \qquad \forall i,j = 1,\ldots,n,$$

$$\sum_{k=1}^{n} \sum_{\ell=1}^{n} \lambda_{k\ell}^{t} \frac{1}{q_k} = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij}^{t+1} \frac{1-q_i}{q_i} \qquad t = 1,\ldots,n-1,$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij}^{n} = \sum_{i=1}^{n} \sum_{j=1}^{n} q_i x_{ij}^{n},$$

Compact Formulation with 3-indices variables ($3F2^K$)

- $x_{ij} = \sum_{t=n-K}^{n} x_{ij}^t$
- $z_{it} = \sum_{j=1}^{n} x_{ij}^t \quad \forall t = n-K+1, \ldots, n$
- λ_{ij}^t

Compact Formulation with 3-indices variables ($3F2^K$)

min $\sum_{t=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij}^{t} d_{ij}$

s.t. p open facilities

one customer, one assignment, one order

only assignments to open facilities

bounds on λ from x and z

right λ sequence

Compact Formulation with 3-indices variables ($3F2^K$)

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$$\sum_{k=1}^{n} \sum_{\ell=1}^{n} \lambda_{k\ell}^{t} \frac{1}{q_k} = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij}^{t+1} \frac{1-q_i}{q_i} \qquad t = 1, \ldots, n-1$$

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{d_{kj}}{q_k} \lambda_{kj}^{n} \geqslant \sum_{j=1}^{n} x_{ij} d_{ij} \qquad i = 1, \ldots, n$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} \frac{1}{q_i} \lambda_{ij}^{t} \leqslant \sum_{k=1}^{n} \sum_{\ell=1}^{n} d_{k\ell} \frac{(1-q_k)}{q_k} \lambda_{k\ell}^{t+1} \qquad t = 1, \ldots, n-1$$

Preprocessing phase

Preprocessing 1:
The optimal value of the pCP problem for $q_{(1)}d_{ij}$ for any $i, j \in N$ yields a lower bound on the optimal PpCP value.
Therefore, $x_{ij}^n = 0$ and $\lambda_{ij}^n = 0 \, \forall i, j \in N$ and $d_{ij}q_i < d^* q_{(1)}$.
Preprocessing 2:
z^*_{p+K}, the optimal value of the (p+K)CP, is a LB of the K-th distance from a site and its corresponding service in the PpCP.
Therefore, $x_{ij}^t = 0$ and $\lambda_{ij}^t = 0$ if $d_{ij} < z^*_{p+K}$, $t = n - K + 1, \dots, n$.
Preprocessing 3:
An upper bound of the $PpCP$, UB_{PpCP}, we have that $x_{ij}^t = 0$ and $\lambda_{ij}^t = 0$, $\forall i, j \in N, t = n - K + 1, \dots, n$, such that, $q_i d_{ij} > UB_{PpCP}$.

Preprocessing phase

Consider the ordered sequence of distances between sites $i, j \in \{1, \dots, n\}$, $0 = d_{(1)} \leqslant \dots \leqslant d_{(G)} = \max_{i,j \in N} \{d_{ij}\}$.

Preprocessing 4: Given h, consider the following problem:

$$
\begin{aligned}
n_U(h) = \max \sum_{(i,j): d_{ij} \geqslant d_{(h)}} & x_{ij}, \\
s.t. \quad \sum_{j=1}^{n} x_{jj} &= p, \\
x_{ij} &\leqslant x_{jj}, & i, j = 1, \dots, n \\
\sum_{j=1}^{n} x_{ij} &= 1, & i = 1, \dots, n \\
\sum_{j': d_{ij'} > d_{ij}} x_{ij'} + x_{jj} &\leqslant 1, & i, j = 1, \dots, n.
\end{aligned}
$$

Then, $x_{ij}^t = 0$ and $\lambda_{ij}^t = 0$ if $d_{ij} = d_{(h)}$ and $n_U(h) < n - t$, $t = n - K + 1, \dots, n$.

Preprocessing phase

Consider the ordered sequence of distances between sites $i, j \in \{1, \dots, n\}$, $0 = d_{(1)} \leqslant \dots \leqslant d_{(G)} = \max_{i,j \in N}\{d_{ij}\}$.

Preprocessing 5: Given h, consider the following subproblem

$$
\begin{aligned}
n_L(h) = \max \sum_{(i,j): d_{ij} \leqslant d_{(h)}} & x_{ij}, \\
s.t. \sum_{j=1}^{n} x_{jj} &= p, \\
x_{ij} &\leqslant x_{jj} && i, j = 1, \dots, n \\
\sum_{j=1}^{n} x_{ij} &= 1, && i = 1, \dots, n \\
\sum_{j' : d_{ij'} > d_{ij}} x_{ij'} + x_{jj} &\leqslant 1, && i, j = 1, \dots, n.
\end{aligned}
$$

Then, $x_{ij}^t = 0$ and $\lambda_{ij}^t = 0$ if $d_{ij} = d_{(h)}$ and $n_L(h) < t - 1$.

Experiment details

- Distance submatrices from ORLIB p-median data.
  http://people.brunel.ac.uk/∼mastjjb/jeb/orlib/files/
- Dimensions: $n \in \{6, 10, 13, 15, 20, 25, 30\}$, $p \in \{2, 3, 5, 8, 10\}$
- Formulations implemented in Xpress 7.7.
- Intel(R) Core(TM) i7-4790K CPU 32 GB RAM.
- Time limit: 2 hours.

Computational results

			Best $4F^K$			$3F^K$			Best $3F^K$			$3F2^K$			Best $3F2^K$		
n	p	k	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap LP	Nodes	Time	Gap LP	Nodes	Time
6	2	2	84.43	136	0.77	74.96	121	0.04	61.61	54	0.07	87.32	84	0.07	60.91	30	**0.05**
10	3	3	91.72	488	53.71	79.47	1911	0.73	34.57	430	**0.25**	91.52	1181	0.32	40.41	261	0.26
10	5	3	92.59	849	65.51	74.25	1287	0.46	52.68	290	0.21	95.18	1543	0.29	54.85	167	**0.20**
13	3	4	96.10	1364	1304.23	89.94	11158	7.98	50.63	1724	**1.06**	94.99	4971	2.57	54.56	1854	2.36
13	5	4	94.90	3427	2309.56	84.16	19230	10.21	47.19	2770	**1.10**	95.54	12308	4.19	49.47	1976	1.70
13	8	4	93.24	1878	1165.70	65.06	1680	1.50	44.78	383	**0.38**	96.71	6568	2.53	44.48	344	0.61
15	3	4	97.84	1572	7172$^{(4)}$	92.39	15588	18.17	54.32	2519	**2.38**	97.25	9933	9.66	58.92	2653	5.63
15	7	4	96.82	2357	7223$^{(5)}$	83.57	33702	23.79	47.95	3653	**1.90**	98.11	37229	22.45	50.33	2195	2.86
15	10	4	93.75	2673	5945$^{(3)}$	59.37	1647	2.44	35.82	325	**0.61**	98.05	9306	7.30	39.27	294	1.12
20	3	5	-	-	-	95.19	94839	405.88	53.91	6649	**15.93**	98.11	50555	216.89	60.54	15458	139.68
20	7	5	-	-	-	88.16	812191	1955.00	40.60	37100	**33.47**	98.10	515055	1353.35	45.61	19148	87.54
20	10	5	-	-	-	82.86	506072	939.36	41.46	11450	**9.35**	98.51	834787	2110.14	45.12	6058	27.24
25	3	6	-	-	-	97.62	478684	6903$^{(3)}$	48.55	24442	**134.86**	98.15	173028	2291.20	58.62	42250	1499.58
25	7	6	-	-	-	-	-	-	42.53	209666	**465.44**	98.26	762834	7201$^{(5)}$	48.22	135555	2112.55
25	10	6	-	-	-	-	-	-	42.10	312288	**582.74**	-	-	-	46.59	80041	1084.64
30	3	7	-	-	-	-	-	-	49.91	111984	**1206.08**	-	-	-	64.26	62352	6994$^{(4)}$
30	7	7	-	-	-	-	-	-	52.93	1063114	**7204**$^{(5)}$	-	-	-	60.60	78951	7210$^{(5)}$
30	10	7	-	-	-	-	-	-	41.03	1148241	**4952**$^{(2)}$	-	-	-	50.41	112441	5970$^{(4)}$

			Percentage of fixed x variables								
n	p	K	Prep. 1	Prep. 2	Prep. 4	Prep. 5	Prep. 3*	All prep.	Prep. 1, 3*, 5	Prep. 1, 2, 3*, 5	Prep. 1, 3*, 4, 5
6	2	2	14.81	7.59	0.00	6.67	27.78	46.67	45.37	46.67	45.37
10	3	3	12.60	3.40	1.00	8.80	38.70	54.55	54.00	54.40	54.15
10	5	3	9.00	3.05	1.00	3.30	46.35	57.50	57.05	57.35	57.20
13	3	4	12.59	2.51	1.42	11.98	35.41	53.75	53.14	53.66	53.23
13	5	4	9.37	1.92	1.66	5.49	47.91	59.76	59.48	59.64	59.60
13	8	4	6.15	1.78	3.64	1.56	59.83	67.69	67.38	67.62	67.46
15	3	4	11.59	1.97	1.03	12.73	36.69	55.08	54.61	54.92	54.77
15	7	4	7.47	1.48	1.21	3.73	53.33	62.49	62.38	62.44	62.44
15	10	4	5.33	1.40	2.92	1.10	64.57	71.00	70.93	71.00	70.93
20	3	5	12.08	1.28	1.03	15.92	34.63	55.11	54.71	54.87	54.95
20	7	5	7.67	0.98	1.25	5.27	55.75	65.26	65.08	65.14	65.19
20	10	5	5.83	0.91	1.40	2.83	62.08	69.17	69.05	69.09	69.13
25	3	6	10.78	1.01	1.05	17.95	34.67	56.46	56.01	56.19	56.28
25	7	6	7.27	0.73	1.06	6.27	57.46	67.15	67.03	67.10	67.08
25	10	6	5.90	0.68	1.23	3.79	62.87	70.10	70.03	70.08	70.05
30	3	7	9.49	0.83	0.99	19.04	34.67	57.11	56.62	56.80	56.94
30	7	7	7.16	0.55	1.02	6.75	56.04	65.61	65.51	65.74	65.58
30	10	7	5.79	0.51	1.03	4.43	62.79	69.99	69.92	69.96	70.08

			Percentage of fixed λ variables								
n	p	K	Prep. 1	Prep. 2	Prep. 4	Prep. 5	Prep. 3*	All prep.	Prep. 1, 3*, 5	Prep. 1, 2, 3*, 5	Prep. 1, 3*, 4, 5
6	2	2	22.22	11.39	0.00	10.00	41.67	68.33	68.06	68.33	68.06
10	3	3	16.80	4.53	1.33	11.73	51.60	72.47	72.00	72.27	72.20
10	5	3	12.00	4.07	1.33	4.40	61.80	76.33	76.07	76.13	76.27
13	3	4	15.74	3.14	1.78	14.97	44.26	66.72	66.42	66.60	66.54
13	5	4	11.72	2.40	2.07	6.86	59.88	74.53	74.35	74.38	74.50
13	8	4	7.69	2.22	4.56	1.95	74.79	84.38	84.23	84.29	84.32
15	3	4	14.49	2.47	1.29	15.91	45.87	68.56	68.27	68.36	68.47
15	7	4	9.33	1.84	1.51	4.67	66.67	78.04	77.98	77.98	78.04
15	10	4	6.67	1.76	3.64	1.38	80.71	88.69	88.67	88.69	88.67
20	3	5	14.50	1.53	1.24	19.10	41.55	65.94	65.65	65.65	65.94
20	7	5	9.20	1.17	1.50	6.32	66.90	78.23	78.09	78.09	78.23
20	10	5	7.00	1.09	1.68	3.40	74.50	82.97	82.86	82.88	82.95
25	3	6	12.58	1.18	1.23	20.94	40.45	65.67	65.34	65.35	65.66
25	7	6	8.48	0.85	1.24	7.32	67.04	78.27	78.20	78.21	78.26
25	10	6	6.88	0.79	1.43	4.43	73.34	81.74	81.70	81.71	81.73
30	3	7	10.84	0.95	1.14	21.77	39.62	65.08	64.71	64.72	65.08
30	7	7	8.18	0.63	1.16	7.71	64.04	74.95	74.87	75.10	74.94
30	10	7	6.62	0.59	1.18	5.07	71.76	79.95	79.90	79.92	80.10

Heuristic approach for PpCP

Variable neighborhood search

- Domínguez, Hansen, Mladenovic, Nickel (1997).
- Puerto, Pérez, García (2014).

n	p	K	Gap	VNS time	Best $F3^K$ time
6	2	2	0.00	0.00	0.07
10	3	3	0.00	0.00	0.25
10	5	3	0.00	0.00	0.21
13	3	4	3.89	0.00	1.06
13	5	4	0.24	0.00	1.10
13	8	4	0.00	0.00	0.38
15	3	4	0.00	0.00	2.38
15	7	4	1.98	0.02	1.90
15	10	4	0.00	0.02	0.61
20	3	5	0.00	0.00	15.93
20	7	5	0.04	0.02	33.47
20	10	5	0.00	0.03	9.35
25	3	6	1.63	0.00	134.86
25	7	6	0.00	0.04	465.44
25	10	6	0.35	0.06	582.74
30	3	7	0.00	0.02	1206.08
30	7	7	0.74	0.07	$7204^{(5)}$
30	10	7	2.37	0.11	$4952^{(2)}$

Conclusions

- Homogeneous probabilities $\longrightarrow$ Ordered p-Median
- 3 MIP formulations:

  Need to improve lower bounds:
  - OMP with $\lambda_t = q_{(n-t+1)} \prod_{i=1}^{n-t}(1 - q_{(i)})$.
- Formulation based on
  - J. R. O'Hanley, M. P. Scaparra, and S. García (2013).