Using a kernel search heuristic to solve a sequential competitive location problem in a discrete space

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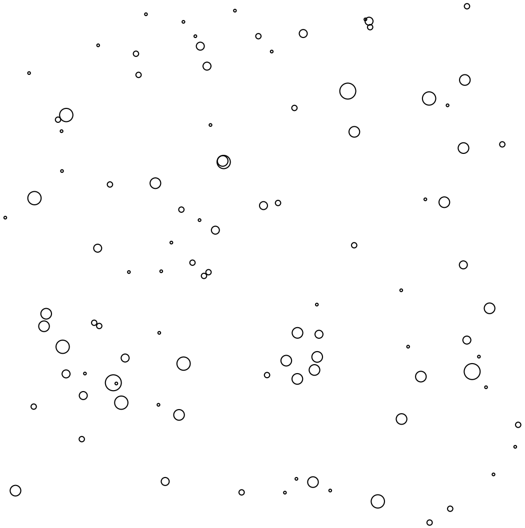
International Workshop on Locational Analysis and Related Problems
Cádiz, January 30-February 1, 2019

The discrete leader-follower location problem



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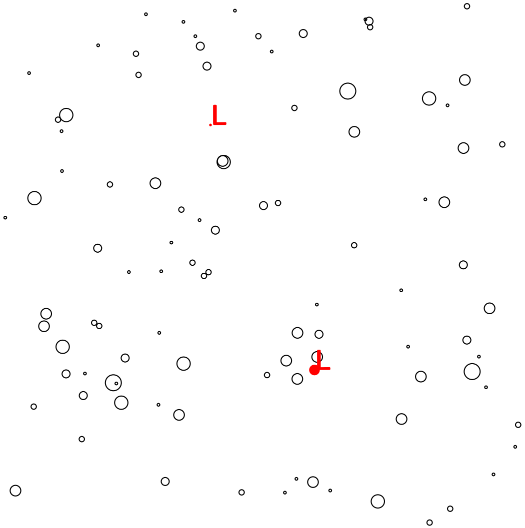

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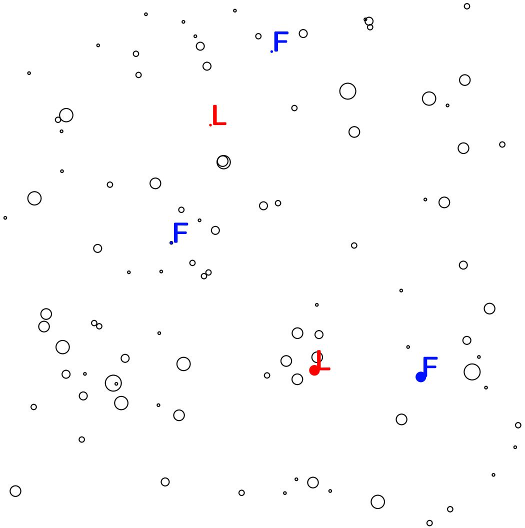


The discrete leader-follower location problem

- Two players, the **leader** (Firm A) and the **follower** (Firm B), **enter the market sequentially** and decide where to **locate their facilities** in order to optimize certain objectives.
  - **Leader's problem**: The leader enters the market first with p facilities and seeks to minimize the maximum market share captured by a future competitor.
  - **Follower's problem**: The follower opens r facilities at the locations that maximize its market share.

$$\downarrow$$

$$\textbf{Bilevel problem} \left\{ \begin{array}{lcl} \text{upper level} & \rightarrow & \text{leader's problem} \\ \text{lower level} & \rightarrow & \text{follower's problem} \end{array} \right.$$

The discrete leader-follower location problem

- **Essential** goods or services.
- **Customer choice rule** defined by an **S-shaped function**. The **binary rule** is a borderline case of this kind of function.
- **Market share**:

  Leader (firm A) with **p facilities** located at $X_p \in L^p$
  Follower (firm B) with **r facilities** located at $Y_r \in L^r$

$$\Big\downarrow \text{Market share}$$

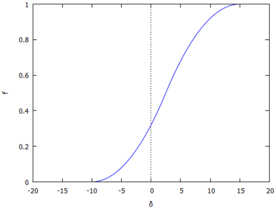
$$W_A = W_T - W_B$$

$$W_B = W_B(X_p, Y_r) = \sum_{k=1}^{n} \underbrace{w_k f_k(\delta_k)}_{\text{demand at } c_k \text{ captured by B}}$$

where $\delta_k = d_{kX_p} - d_{kY_r} = d_{kA} - d_{kB}$.

The discrete leader-follower location problem

$$f_k(\delta) = \begin{cases} 0 & \text{if} \quad \delta \le a_k \\ 2\left(\dfrac{\delta - a_k}{b_k - a_k}\right)^2 & \text{if} \quad a_k < \delta \le \dfrac{a_k + b_k}{2} \\ 1 - 2\left(\dfrac{\delta - b_k}{b_k - a_k}\right)^2 & \text{if} \quad \dfrac{a_k + b_k}{2} < \delta \le b_k \\ 1 & \text{if} \quad \delta > b_k \end{cases}$$



where $a_k \le 0 < b_k, \; k \in [1..n]$.

Linear formulations: follower's problem

$$W_B(X_p, Y_r) = \max_{Y \in L^r} W_B(X_p, Y) \tag{1}$$

$$\downarrow$$

$$\begin{array}{ll}
\max \sum_{i=1}^{m} \sum_{k=1}^{n} h_{ki} z_{ki} & \\
\sum_{i=1}^{m} y_i = r & \\
\sum_{i=1}^{m} z_{ki} \leq 1 & k \in [1..n] \\
z_{ki} \leq y_i & i \in [1..m], \; k \in [1..n] \\
z_{ki}, \; y_i \in \{0, 1\} & i \in [1..m], \; k \in [1..n].
\end{array}$$

where:

$$y_i = \begin{cases} 1 & \text{if the follower opens a facility at point } l_i \\ 0 & \text{otherwise} \end{cases}$$

$$z_{ki} = \begin{cases} 1 & \text{if customer at } c_k \text{ visits a facility at point } l_i \\ 0 & \text{otherwise.} \end{cases}$$

$$h_{ki} = w_k f_k(\delta_{ki}) \text{ and } \delta_{ki} = d_{kX_p} - d_{ki}.$$

Linear formulations: leader's problem

$$\min_{X \in L^p} \underbrace{\max_{Y \in L^r} W_B(X, Y)}_{S(X)} . \qquad (2)$$

$$\downarrow$$

$$\begin{array}{ll}
\min \ W & \\
\sum_{i=1}^{m} x_i = p & \\
\sum_{i=1}^{m} \sum_{k=1}^{n} h^j_{ki} u_{ki} \leq W & j \in J \\
\sum_{i=1}^{m} u_{ki} = 1 & k \in [1..n] \\
u_{ki} \leq x_i & i \in [1..m], \ k \in [1..n] \\
u_{ki}, \ x_i \in \{0, 1\} & i \in [1..m], \ k \in [1..n].
\end{array}$$

where:

$$x_i = \begin{cases} 1 & \text{if the leader opens a facility at point } l_i \\ 0 & \text{otherwise} \end{cases}$$

$$u_{ki} = \begin{cases} 1 & \text{if customer at } c_k \text{ visits a facility at point } l_i \\ 0 & \text{otherwise} \end{cases}$$

and, for $i \in [1..m]$, $k \in [1..n]$ and $j \in J = [1..\binom{m}{r}]$,

$$h^j_{ki} = w_k f_k(\delta^j_{ki}) \qquad \delta^j_{ki} = d_{ki} - d_{kY_j} . \qquad (3)$$

Solution procedure: matheuristic procedure

Initialization:

s feasible leader solutions X_i, i=1,…,s
Solve the follower's problem for X_i, i=1,…,s
Upper bound of W*: $\overline{W} = \min_i S(X_i)$
Initial optimal solution: X* with S(X*) = $\overline{W}$
Initial lower bound: $\underline{W}$
Initial family of follower solutions: F
F = $\{Y(X_i)\}_{i=1,\dots,s}$

Iterations:

<u>Solve the leader's problem</u> using the family F : X, S_F (X)

If S_F (X) > $\underline{W}$ do $\underline{W}$ = S_F (X)

Solve the follower's problem for X

If S(X) < $\overline{W}$ do $\overline{W}$=S(X) and X*=X

$\underline{W}=\overline{W}$

YES

STOP

NO

New family of followers: F = F U {Y(X)}

KERNEL SEARCH ALGORITHM:
[Guastaroba and Speranza (2012, 2014), Mansini and Speranza (1999)]

1. Solve the LP relaxation
2. Build the initial kernel and a sequence of buckets
3. Solve a MILP problem on the initial kernel
4. Repeat the following until a certain number of buckets have been analysed:
(a) Solve the MILP problem on the current kernel plus a bucket
(b) Update the current kernel

Solution procedure: exact procedure

Algorithm. Exact procedure to solve the leader-follower problem

1. *Initialization.*
   1.1 Select s feasible leader solutions $X_i, \; i = 1, \ldots, s$.
   1.2 Solve the follower problem for X_i, $i = 1, \ldots, s$.
   1.3 Calculate an upper bound $\overline{W}$ of the optimum W^*
   $\overline{W} = \min_i S(X_i)$.
   1.4 Let $X^* = X$ with $S(X) = \overline{W}$.
   1.5 Let $\mathcal{F} = \{Y_i\}_{i=1}^{s}$ be the selected family of good feasible solutions for the follower.
   1.6 Set a lower bound $\underline{W}$.
2. *Iterations.* Repeat, until a stop rule condition is satisfied.
   2.1 Do $i = i + 1$.
   2.2 Solve the leader problem using the family $\mathcal{F}$ of follower solutions.
   Let X be the optimal solution obtained.
      2.0.1 If the optimal value obtained $S_{\mathcal{F}}(X)$ verifies $S_{\mathcal{F}}(X) > \underline{W}$ then do $\underline{W} = S_{\mathcal{F}}(X)$.
      2.0.2 If $\underline{W} = \overline{W}$, then $W^* = \underline{W} = \overline{W}$ is the optimal value and X^* is the optimal location set for the leader.
   2.3 Solve the follower problem for X.
      2.0.1 If $S(X) < \overline{W}$ then set $\overline{W} = S(X)$ and $X^* = X$.
      2.0.2 If $\underline{W} = \overline{W}$, then $W^* = \underline{W} = \overline{W}$ is the optimal value and X^* is the optimal location set for the leader.

Set $\mathcal{F} = \mathcal{F} \cup \{Y(X)\}$, where $Y(X)$ is the solution to the follower problem.

Solution procedure: matheuristic procedure

Algorithm. Kernel search heuristic to solve the leader-follower problem

1. **Solve the LP-relaxation** of the leader problem restricted to family $\mathcal{F}$.
2. **Build the initial kernel** $(K, U(K))$ where
   $K = \{x_i$ such that $x_i \neq 0$ for the LP-relaxation solution $\}$
   $U(K) = \{u_{ki}$ such that $x_i \neq 0\}$.
3. Sort the other variables according to a criterion based on their reduced costs and build a sequence of disjoint variable buckets with the same given length, except the last bucket which can have a smaller length.
4. **Solve the MILP on the initial kernel**.
5. Repeat until a certain number of buckets have been analyzed.
   5.1 Let $(\hat{K}, U(\hat{K}))$ be the current kernel plus a bucket.
   5.2 **Solve the MILP on** $(\hat{K}, U(\hat{K}))$ with two constraints:
      5.2.1 Set an upper bound on the objective function value.
      5.2.2 At least one facility of the current bucket must be selected.
   Let K_h^+ be the set of facilities of the current bucket that are selected in the feasible solution.
   Let K_h^- be the set of facilities belonging to the current kernel that have not been selected in t of the previously solved restricted problems since they have been added.
   5.3 **Update the current kernel**:
   $\hat{K} \leftarrow \hat{K} \cup \; K_h^+ \setminus K_h^-$.

Computational example ($5 \times 4 \times 20 = 480$ scenarios)

(p, r)	**Customer choice rule**
(2,2)	Binary $\mu = 0.5$
(2,3)	S_k-function $(f_k(\delta))$, $k = 1, ..., 5$, where:
(3,2)	$a_k = -\alpha_k \times \rho$, $b_k = \beta_k \times \rho$ with $\rho =$ average $\lvert d_{ki} - d_{kj} \rvert$, $k \in K$, $i, j \in I$, $i \neq j$
(5,5)	$\alpha_k = \beta_k = 0.01, 0.05, 0.10, 0.25$ and α_k, β_k randomly generated with uniform distribution in $[0.01, 0.25]$
	Kernel search parameters
	Initial kernel size: $q=$ number of nonzero variables in solution to relaxed problem
	Bucket size: $lbuck = q$
	Number of buckets: $nbuck = \lceil \frac{m-q}{lbuck} \rceil$
	Maximum number of buckets investigated: $\overline{nbuck} = \min\{nbuck, 3\}$
	Parameter for location variables elimination from the kernel: $s = 2$
	Demand configurations
	20 instances (Alekseeva et al, 2010): $C = L$, $n = m = 100$.
	10 spatial configurations. Instances 1-10: $w_k = 1$, $\forall k$.
	Instances 11-20: w_k generated by a uniform distribution in $[0, 200]$.
	Initial solutions
	p-median, solution obtained via the alternating algorithm.
	Initial family: $(r \vert X)$- medianoid for X the p-median, r-median, and $(r \vert X)$-medianoid for X solution via alternating heuristic.

Results. Binary rule

Table: Binary rule. Average values for different procedures and scenarios

Scenario p	r	Instances	W^*	ER (%)	IT	ITO	T (s)	TO (s)	ΔW^* (%)
					Exact				
2	2	1-10	52.30	0	10.00	4.20	**77.69**	**22.99**	
2	2	11-20	5370.35	0	14.70	9.20	86.92	44.63	
2	3	1-10	72.05	0	41.90	20.70	1374.91	558.50	
2	3	11-20	7273.95	0	51.40	16.20	1562.71	170.01	
3	2	1-10	37.20	0	12.00	6.60	95.30	41.42	
3	2	11-20	3756.30	0	15.20	10.40	134.86	100.32	
5	5	1-10	53.45	0.20	53.10	15.80	3844.15	202.58	
5	5	11-20	5386.90	0.11	75.20	52.10	**13434.71**	**5963.88**	
					RKS				
2	2	1-10	52.30		8.00	4.10	**12.59**	6.48	0
2	2	11-20	5402.90		18.50	6.50	32.99	10.22	0.34
2	3	1-10	72.60		29.50	3.80	46.80	**5.82**	0.55
2	3	11-20	7375.10		17.30	5.60	31.69	8.26	0.97
3	2	1-10	37.45		10.20	7.40	16.78	12.36	0.25
3	2	11-20	3771.35		29.60	6.80	59.12	11.98	0.15
5	5	1-10	53.45		33.80	15.30	91.04	36.31	0
5	5	11-20	5403.00		62.70	38.30	**277.39**	**161.07**	0.16
					KS				
2	2	1-10	52.30		10.40	4.50	**44.06**	**14.70**	0
2	2	11-20	5378.00		14.00	9.10	67.07	37.75	0.09
2	3	1-10	72.35		34.10	15.20	196.47	68.62	0.30
2	3	11-20	7301.95		60.80	18.10	491.80	83.73	0.27
3	2	1-10	37.20		12.60	6.90	65.46	35.81	0
3	2	11-20	3756.30		24.20	8.90	138.29	56.09	0
5	5	1-10	53.35		49.80	19.10	1132.37	162.79	-0.10
5	5	11-20	5394.35		77.20	39.20	**3721.35**	**1308.11**	0.07

Results. Binary rule

Table: Binary rule $p = r = 5$

	Exact						RKS						KS					
In	W^*	ER	IT	ITO	T	TO	W^*	IT	ITO	T	TO	ΔW^*	W^*	IT	ITO	T	TO	ΔW^*
1	53	0	54	10	1596.80	52.01	53.50	37	22	87.02	45.80	0.50	53	52	19	526.64	93.23	0
2	53	0	38	11	755.88	59.64	53	25	9	51.22	13.49	0	53	36	10	334.20	36.98	0
3	55	1	100	19	**17017.02**	350.61	55	57	8	**198.08**	12.81	0	55	100	7	**5303.26**	26.05	0
4	55	1	100	9	12736.50	44.94	54	45	43	147.93	**138.81**	-1	54	74	13	2092.76	75.71	-1
5	53	0	39	20	1111.03	288.02	53	28	16	66.33	31.42	0	53	40	25	644.93	276.44	0
6	53	0	38	25	854.99	351.65	53	31	31	71.86	71.86	0	53	39	33	425.02	327.04	0
7	53	0	42	16	1336.56	160.92	53	22	6	**45.75**	9.19	0	53	32	30	**321.34**	285.33	0
8	53	0	34	12	**609.12**	80.75	53	26	18	66.51	39.72	0	53	33	11	412.49	64.09	0
9	53	0	45	0	1556.72	**0**	53	31	0	93.56	**0**	0	53	47	0	782.92	**0**	0
10	53.50	0	41	36	866.89	**637.27**	54	36	0	82.11	0	0.50	53.50	45	43	480.15	**443.05**	0
Av	53.45	0.20	53.10	15.80	3844.15	202.58	53.45	33.80	15.30	91.04	36.31	0	53.35	49.80	19.10	1132.37	162.79	-0.10
11	4550	0	29	13	**208.38**	**52.54**	4550	21	11	**41.99**	**19.31**	0	4550	32	12	**184.18**	**47.72**	0
12	5698	0	93	64	16127.27	4752.99	5716	100	38	418.18	113.62	0.17	5698	95	28	4475.34	287.91	0
13	5136	0.41	100	91	31207.86	22652.04	5136	93	88	**626.18**	**571.33**	0	5136	100	91	8441.67	**6636.40**	0
14	5354.50	0	86	37	7650.40	826.97	5391.50	86	12	436.62	23.54	0.37	5354.50	87	42	4369.49	622.92	0
15	5669.50	0.35	100	96	27970.40	**24412.06**	5669.50	69	69	284.79	284.79	0	5671	100	27	3883.79	268.99	0.01
16	5173.50	0	69	51	3317.14	1625.56	5173.50	52	20	165.44	42.58	0	5173.50	72	21	1894.35	214.80	0
17	6046	0	73	48	4757.83	1517.71	6046	46	28	133.96	65.74	0	6046	75	47	2025.60	762.42	0
18	5153	0	67	43	3385.22	1022.36	5191	56	31	183.57	75.51	0.40	5153	73	32	2490.84	391.28	0
19	5696	0.35	100	43	**39215.41**	2269.38	5696	72	70	409.48	384.59	0	5769	100	68	**9124.16**	3691.15	0.70
20	5392.50	0	35	35	507.18	507.18	5460.50	32	16	73.70	29.73	0.66	5392.50	38	24	324.08	157.50	0
Av	5386.90	0.11	75.20	52.10	13434.71	5963.88	5403	62.70	38.30	277.39	161.07	0.16	5394.35	77.20	39.20	3721.35	1308.11	0.07

Results. S-function

Table: S-shaped function. Average values for exact procedure and different scenarios

Scenario				W^*	ER	IT	ITO	T	TO
S	p	r	Instances		(%)			(s)	(s)
1	2	2	1-10	52.31	0	10.3	3.2	**83.58**	**16.72**
1	2	2	11-20	5357.69	0	13.9	7.3	84.64	32.23
1	2	3	1-10	71.99	0	43.8	19.7	1477.82	482.15
1	2	3	11-20	7269.08	0	48.1	23.2	1296.66	399.50
1	3	2	1-10	37.15	0	11.6	8.9	93.39	68.51
1	3	2	11-20	3750.31	0	15.1	8.2	136.78	67.04
1	5	5	1-10	53.24	0.08	54.8	30.3	5006.03	1477.84
1	5	5	11-20	5382.26	0.14	74.3	36.1	**10875.83**	**2773.17**
2	2	2	1-10	52.04	0	9.1	2.7	**73.20**	**15.92**
2	2	2	11-20	5341.50	0	13.5	8.5	83.60	46.03
2	2	3	1-10	71.91	0	42.1	21.3	1337.61	497.47
2	2	3	11-20	7252.09	0	47.6	26.1	1248.92	592.79
2	3	2	1-10	36.94	0	11.7	9.7	117.37	97.67
2	3	2	11-20	3742.80	0	15.4	8.3	154.10	79.43
2	5	5	1-10	52.85	0	48.6	29.8	4529.22	2056.45
2	5	5	11-20	5324.36	0	64.2	46.8	**10299.19**	**4458.78**
3	2	2	1-10	51.88	0	8.5	6.3	70.02	45.65
3	2	2	11-20	5298.78	0	11.5	8	**64.71**	**39.83**
3	2	3	1-10	71.47	0	36.7	14.1	944.61	278.01
3	2	3	11-20	7204.36	0	43.2	15.4	1062.78	311.95
3	3	2	1-10	36.92	0	10.7	9	114.13	100.80
3	3	2	11-20	3726.01	0	14	9.9	138.41	104.95
3	5	5	1-10	52.51	0	39.8	19.4	1582.85	241.18
3	5	5	11-20	5269.61	0	50.8	35	**4942.89**	**2972.60**
4	2	2	1-10	51.47	0	5.4	1.4	35.25	**8.61**
4	2	2	11-20	5180.06	0	7.1	4.9	**27.88**	17.84
4	2	3	1-10	70.07	0	23.1	13	418.54	142.75
4	2	3	11-20	6953.81	0	20.9	11.4	260.61	137.70
4	3	2	1-10	36.76	0	8.4	5.3	71.31	47.21
4	3	2	11-20	3662.33	0	8.9	6.6	58.96	45.80
4	5	5	1-10	51.42	0	18	12.5	228.49	112.58
4	5	5	11-20	5158.16	0	23.9	18.5	**655.49**	**413.79**
5	2	2	1-10	51.72	0	8.1	2.9	59.33	**15.27**
5	2	2	11-20	5232.43	0	9.9	6.2	**50.62**	27.95
5	2	3	1-10	70.96	0	33.6	15.3	862.11	215.15
5	2	3	11-20	7083.66	0	33.7	13	668.86	213.38
5	3	2	1-10	37.39	0	13.3	9	135.07	98.51
5	3	2	11-20	3715.43	0	11.3	8.3	86.36	60.85
5	5	5	1-10	52.91	0.01	51.3	40.3	3763.70	1965.34
5	5	5	11-20	5300.14	0	59.6	33.9	**6268.37**	**1478.33**

Results. S-function

Table: S-shaped function. Average values for RKS and different scenarios

Scenario				W^*	IT	ITO	T	TO	ΔW^*
S	p	r	Instances				(s)	(s)	(%)
1	2	2	1-10	52.48	25.9	2.7	46.26	4.37	0.17
1	2	2	11-20	5373.28	27.4	5.9	45.47	9.29	0.17
1	2	3	1-10	72.87	65.8	2.1	118.18	**3.49**	0.88
1	2	3	11-20	7314.96	58.3	4.6	137.75	8.24	0.44
1	3	2	1-10	37.38	9.2	7.1	**16.02**	12.01	0.23
1	3	2	11-20	3779.57	19.9	6.6	45.81	12.67	0.32
1	5	5	1-10	53.46	48	18.8	157.25	49.05	0.22
1	5	5	11-20	5386.83	56	27.2	**256.90**	**94.61**	0.04
2	2	2	1-10	52.44	24.2	2.1	42.71	**3.54**	0.40
2	2	2	11-20	5375.49	27.5	6.1	46.56	9.97	0.35
2	2	3	1-10	72.84	55	2.3	117.70	3.70	0.93
2	2	3	11-20	7330.01	41.5	5.8	93.05	9.55	0.75
2	3	2	1-10	37.08	17.8	5.9	**27.52**	9.95	0.14
2	3	2	11-20	3770.08	20.9	5.9	41.34	11.74	0.29
2	5	5	1-10	52.94	39.6	19.4	116.24	56.09	0.09
2	5	5	11-20	5334.03	53.4	31.4	**227.21**	**97.55**	0.10
3	2	2	1-10	51.98	5.6	3.1	**9.67**	**5.40**	0.11
3	2	2	11-20	5322.39	26.8	6.5	46.96	11.23	0.24
3	2	3	1-10	72.19	46	4.1	99.63	6.72	0.72
3	2	3	11-20	7259.54	40.7	6	69.32	12.64	0.53
3	3	2	1-10	37.21	35.3	5.4	71.67	9.63	0.29
3	3	2	11-20	3750.25	55.3	4.4	116.82	7.59	0.26
3	5	5	1-10	52.74	43.9	19.5	136.43	49.21	0.23
3	5	5	11-20	5282.84	51.6	20.2	**198.58**	**52.95**	0.13
4	2	2	1-10	51.55	23.2	0.5	41.49	**0.98**	0.08
4	2	2	11-20	5206.83	24.7	2.3	**37.43**	4.10	0.28
4	2	3	1-10	70.77	55.5	4	105.25	7.22	0.70
4	2	3	11-20	6985.38	45.2	3.4	74.28	5.72	0.31
4	3	2	1-10	36.87	26	4.3	53.49	7.48	0.11
4	3	2	11-20	3693.58	26.2	4.9	44.33	9.07	0.33
4	5	5	1-10	51.54	20.3	7.2	44.42	13.92	0.12
4	5	5	11-20	5170.32	27.2	13.1	**74.93**	**36.75**	0.11
5	2	2	1-10	51.97	24.7	3	41.51	**5.32**	0.25
5	2	2	11-20	5255.83	17.5	5.5	**27.84**	9.15	0.27
5	2	3	1-10	71.73	74.1	5.1	153.34	8.90	0.77
5	2	3	11-20	7136.72	38.3	6.1	102.34	11.00	0.51
5	3	2	1-10	37.71	45	4.3	105.03	7.49	0.32
5	3	2	11-20	3719.57	18.6	7	36.40	12.83	0.04
5	5	5	1-10	53.13	47.9	20.2	178.04	58.02	0.22
5	5	5	11-20	5315.08	55.6	25.2	**237.51**	**70.85**	0.15

Results. S-function

Table: S-shaped function. Average values for KS and different scenarios

S	p	r	Instances	W^*	IT	ITO	T (s)	TO (s)	ΔW^* (%)
1	2	2	1-10	52.31	17.8	4.6	137.10	**17.13**	0
1	2	2	11-20	5357.69	22	7.9	101.69	28.96	0
1	2	3	1-10	72.06	30.3	16.1	173.25	81.25	0.07
1	2	3	11-20	7284.31	48.8	18.8	338.53	103.03	0.15
1	3	2	1-10	37.15	11.9	9.1	**63.12**	43.21	0
1	3	2	11-20	3750.31	14.2	8	87.77	45.25	0
1	5	5	1-10	53.22	53	31.7	1088.83	505.17	-0.01
1	5	5	11-20	5385.52	79.4	48.9	**3454.97**	**1938.09**	0.03
2	2	2	1-10	52.04	8.8	3.4	**34.59**	**10.64**	0
2	2	2	11-20	5341.50	12.7	8.1	55.73	31.67	0
2	2	3	1-10	72.17	42.4	11.5	302.18	58.89	0.26
2	2	3	11-20	7295.20	39.3	19.6	248.16	105.53	0.42
2	3	2	1-10	36.96	12.1	10	71.10	61.72	0.02
2	3	2	11-20	3742.80	14.7	9.1	108.16	70.64	0
2	5	5	1-10	52.85	48.5	35.4	1109.14	859.44	0
2	5	5	11-20	5326.53	63.7	46.5	**2452.67**	**1603.04**	0.02
3	2	2	1-10	51.89	8.4	6.4	**36.48**	**25.68**	0.01
3	2	2	11-20	5299.50	19.6	7.7	97.18	29.16	0.01
3	2	3	1-10	71.70	36.1	13	204.94	69.23	0.23
3	2	3	11-20	7210.68	55.4	18	395.82	122.43	0.06
3	3	2	1-10	36.93	10.4	7.6	55.09	40.34	0.004
3	3	2	11-20	3726.01	13.6	10	86.27	60.90	0
3	5	5	1-10	52.51	39.5	25.3	644.34	270.76	0
3	5	5	11-20	5269.61	49.7	29.7	**1350.96**	**576.71**	0
4	2	2	1-10	51.47	4.8	1.3	**17.98**	**5.01**	0
4	2	2	11-20	5184.92	6.5	4	23.02	13.32	0.06
4	2	3	1-10	70.18	27.8	14.1	128.90	63.59	0.11
4	2	3	11-20	6957.10	35	10	171.24	49.06	0.03
4	3	2	1-10	36.76	8	4.7	40.69	21.88	0.004
4	3	2	11-20	3662.33	8.7	6.7	42.88	33.73	0
4	5	5	1-10	51.42	17	12.7	115.70	78.15	0
4	5	5	11-20	5158.16	32	20.1	**359.33**	**233.27**	0
5	2	2	1-10	51.72	7.4	3.1	**31.54**	**10.39**	0
5	2	2	11-20	5232.43	9.3	6.3	37.11	24.46	0
5	2	3	1-10	71.00	49.7	11.6	343.77	49.48	0.04
5	2	3	11-20	7116.21	48.7	7.9	341.11	34.89	0.31
5	3	2	1-10	37.39	21.7	8.3	119.28	49.16	0
5	3	2	11-20	3715.85	20	8.4	102.30	50.32	0.004
5	5	5	1-10	52.91	50.6	40.2	1189.93	667.31	0
5	5	5	11-20	5300.14	60	36.7	**1919.55**	**712.13**	0

Results. S-function

Table: Scenario $S = p = r = 5$

	Exact						RKS						KS					
In	W^*	ER	IT	ITO	T	TO	W^*	IT	ITO	T	TO	ΔW^*	W^*	IT	ITO	T	TO	ΔW^*
1	52.58	0	48	40	1132.92	756.71	52.72	30	12	74.52	23.53	0.13	52.58	48	40	491.20	364.03	0
2	51.86	0	26	16	340.94	133.16	51.86	19	12	**38.78**	22.11	0	51.86	24	15	155.01	**73.45**	0
3	53.44	0	74	52	11326.70	**5605.90**	53.44	56	45	236.36	**175.84**	0	53.44	72	46	2704.42	1105.75	0
4	52.59	0	44	44	1660.90	1660.90	52.59	36	36	120.20	120.20	0	52.59	43	43	702.50	702.50	0
5	54.42	0.09	100	49	**12086.81**	2012.61	54.45	55	14	232.68	33.63	0.04	54.42	100	54	**4376.30**	1258.45	0
6	52.08	0	15	13	**132.13**	**106.11**	53.03	28	4	68.30	**6.50**	0.96	52.08	18	15	**106.87**	81.48	0
7	53.65	0	77	74	5754.94	5247.34	54.41	100	6	**447.93**	10.13	0.75	53.65	81	80	1861.54	**1819.98**	0
8	52.36	0	28	27	331.78	308.82	52.36	22	21	50.64	47.86	0	52.36	28	27	262.74	247.45	0
9	53.03	0	56	53	3477.90	3017.05	53.19	100	19	418.93	48.33	0.16	53.03	49	47	665.44	621.70	0
10	53.06	0	45	35	1391.99	804.82	53.24	33	33	92.08	92.08	0.18	53.06	43	35	573.27	398.33	0
Av	52.91	0.01	51.3	40.3	3763.70	1965.34	53.13	47.9	20.2	178.04	58.02	0.22	52.91	50.6	40.2	1189.93	667.31	0
11	4332.52	0	58	51	1083.25	830.34	4332.52	38	34	90.06	78.10	0	4332.52	56	48	546.23	431.34	0
12	5708.44	0	95	48	20713.20	3379.20	5708.44	100	39	513.99	124.60	0	5708.44	92	47	**5654.79**	1124.12	0
13	4816.27	0	21	6	**313.87**	**29.58**	4816.27	18	4	**39.81**	**6.87**	0	4816.27	21	6	**163.90**	**23.51**	0
14	5365.27	0	49	17	2374.87	171.90	5432.16	42	26	115.87	61.82	0.67	5365.27	49	14	677.77	90.39	0
15	5461.23	0	53	53	2290.71	2290.71	5529.09	48	15	170.03	35.07	0.66	5461.23	58	58	1400.19	1400.19	0
16	5203.66	0	59	28	2742.44	501.57	5203.66	100	19	459.73	42.71	0	5203.66	53	28	713.99	241.24	0
17	5929.46	0	68	47	3688.91	1507.28	5944.10	53	50	194.58	**176.13**	0.13	5929.46	70	36	2250.47	526.13	0
18	5168.43	0	66	60	7107.98	**5580.30**	5168.43	42	38	147.42	125.54	0	5168.43	70	61	2003.68	1524.65	0
19	5625.42	0	92	18	**21851.00**	432.48	5625.42	89	11	**583.20**	24.65	0	5625.42	97	58	5468.86	**1707.57**	0
20	5390.68	0	35	11	517.48	59.97	5390.68	26	16	60.37	33.01	0	5390.68	34	11	315.59	52.12	0
Av	5300.14	0	59.6	33.9	6268.37	1478.33	5315.08	55.6	25.2	237.51	70.85	0.15	5300.14	60	36.7	1919.55	712.13	0

Conclusions

- The heuristics provide good objective values, with time requirements that are lower than those of the exact procedure. This advantage is more significant in cases where the times consumed by the exact algorithm are high.
- In particular, for the scenarios with the highest time consumption (S=p=r=5, instances 11-20):
  - The exact procedure reaches the optimum for all scenarios and the average time values are $T = 6268.37$ and $TO = 1478.33$.
  - For the RKS heuristic the average times are $T = 237.51$ and $TO = 70.85$, and the average ΔW^* is 0.15.
  - KS reaches the optimum for all instances ($\Delta W^* = 0$) with average times of $T = 1919.55$ and $TO = 712.13$.
- A deeper analysis could provide more information about the performance of the algorithms and suggest areas for improvement.
- Other lines of research:
  - It would be useful to apply the proposed solution procedure with different customer choice rules and demand assumptions, and to search for good criteria to sort variables at the bucket-building stage.
  - Comparison with other heuristic procedures.
  - Analysis with different parameter values.