A Branch-and-Price Algorithm for the Vehicle Routing Problem with Stochastic Demands, Optimal Restocking Policy, and Probabilistic Duration Constraints

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Capacitated Vehicle Routing Problem (VRP)

Input:

- $1 + n$ locations: 0 depot and n customers,
- There is a **known** travel cost (duration) between 2 locations,
- There are m identical vehicles with a **known** capacity Q,
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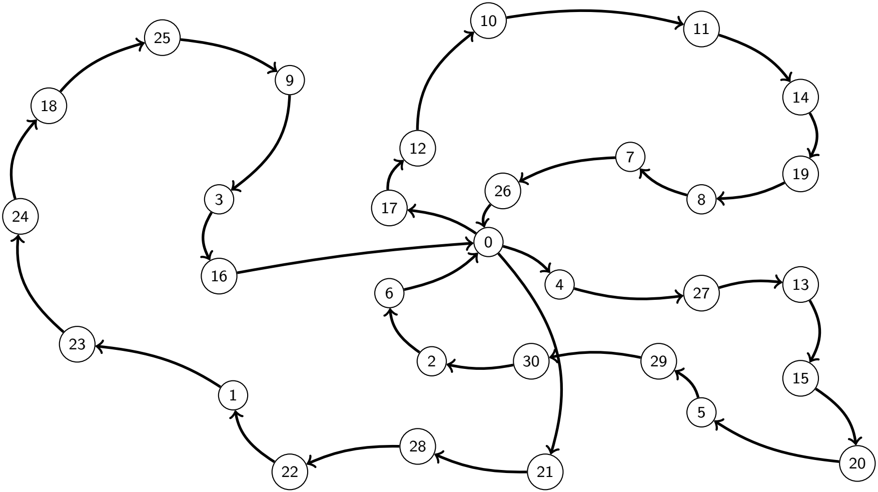
See e.g. the book by Toth & Vigo "Vehicle Routing: problems, methods, and applications" SIAM Series on Optimization 2014.

E031-09h: Optimal solution of VRP: length = 358



VRP with stochastic demands (VRP-SD)

Input:

- Now each customer i has a demand ξ_i which is a random number with a known probability distribution
- The vehicle must go-and-return to the depot when it fails to serve a customer (split demand).

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- These random numbers are independent.

We also assume restocking actions (i.e. preventive returns)

We must **CLUSTER** & **ROUTES** again, but also find a threshold value τ_i for each customer so the driver will go to the depot before the next customer if the load of the vehicle exceeds that threshold.

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- Failure returns: The vehicle must go-and-return to the depot when it fails serving a customer
- Preventive returns: when the current load is over a value (to be determined), the vehicle does not continue the a-priori route, but returns to the depot and after it will resume the a-priori route.

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We must design the a-priori solution to minimize the **expected** total travel cost of the a-posteriori solution: $\sum_r \mathbb{E}[T_r]$.

(see Louveaux & Salazar (2018))

VRP-SD: probabilistic duration constraints

The duration of a route r is a random variable T_r.

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Literature:
Mendoza, Rousseau and Villegas, "A hybrid metaheuristic for the vehicle routing problem with stochastic demand and duration constraints" J. of Heuristics 22 (2016) 539–566.

Expected duration of a given route r

Let us assume that the route is $0, 1, \ldots, n_r, 0$.

The initialization stage defines $f_r(q) = c_{n_r,0}$ for $q = 0, 1, \ldots, Q$.

Assuming $0 \le \xi_j \le Q$, and using $i = j - 1$, the recursion defines

$$f_i(q) = \min \left\{ \quad c_{i,0} + c_{0,j} + \sum_{k=0}^{Q} f_j(Q-k) \cdot p_j^k \quad ; \right.$$

$$\left. c_{i,j} + \sum_{k \le q} f_j(q-k) \cdot p_j^k + \sum_{k > q} \left(c_{j,0} + c_{0,j} + f_j(q+Q-k) \right) \cdot p_j^k \quad \right\}$$

for $j = n_r, \ldots, 2$ and $q = 0, 1, \ldots, Q$.

The expected penalty of the route is then

$$\mathbb{E}[T_r] := c_{0,1} + \sum_{k=0}^{Q} f_1(Q-k) \cdot p_1^k$$

Expected duration of a given route r

Let us assume that the route is $0, 1, \dots, n_r, 0$.

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Without assuming $\xi_j \leq Q$, we need to use:

$$\Gamma_{k,q} = \left\lceil \frac{k-q}{Q} \right\rceil$$

the number of go-and-return to 0 when the demand to serve is k and the remaining load is q $(0 \leq q \leq Q)$.
The recursion is

$$f_i(q) = \min \Bigg\{ c_{i,0} + c_{0,j} + \sum_{k=0}^{\infty} (c_{i,0} + c_{0,i}) \Gamma_{k,q} + f_j(Q + Q\Gamma_{k,q} - k) \cdot p_j^k;$$

$$c_{i,j} + \sum_{k=0}^{\infty} \big((c_{i,0} + c_{0,i}) \Gamma_{k,q} + f_j(q + Q\Gamma_{k,q} - k) \big) \cdot p_j^k \qquad \Bigg\}$$

for $j = n_r, \dots, 2$ and $q = 0, 1, \dots, Q$, and using $i = j - 1$.

The branch-and-price approach

$$
\begin{aligned}
\text{minimize} \quad & \sum_{r \in \mathcal{R}} \mathbb{E}[T_r]\theta_r \,, \\
\text{subject to} \quad & \sum_{r \in \mathcal{R}} a_{ir}\theta_r = 1 \qquad \forall i \in \{1, \ldots, N\} \,, \qquad (1) \\
& \theta_r \in \{0, 1\}
\end{aligned}
$$

where $\mathcal{R}$ is the set of all feasible a priori routes and a_{ir} is a binary value that indicates whether i is visited by the a priori route r.

Checking the feasibility of a route r: Strategy I

Using the Cantelli's inequality (one-sided Chebyshev's bound):

$$\mathbb{P}[T_r > D] = \mathbb{P}\big[T_r - \mathbb{E}[T_r] > D - \mathbb{E}[T_r]\big] \leq \frac{\mathbb{V}(T_r)}{\mathbb{V}(T_r) + (D - \mathbb{E}[T_r])^2}$$

where $\mathbb{V}(T_r)$ can be computed using the law of total variance:

$$\mathbb{V}(T_r^{i,q}) = \sum_{k=0}^{\infty} \mathbb{V}(T_r^{i,q}|\xi_j = k) \cdot p_j^k + \sum_{k=0}^{\infty} \mathbb{E}[T_r^{i,q}|\xi_j = k]^2 \cdot (1 - p_j^k) \cdot p_j^k$$
$$- 2 \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} \mathbb{E}[T_r^{i,q}|\xi_j = k] \cdot p_j^k \cdot \mathbb{E}[T_r^{i,q}|\xi_j = l] \cdot p_j^l \,.$$

being $\mathbb{V}(T_r^{i,q}|\xi_j = k) = \mathbb{V}(T_r^{j,q'})$ with

$$q' = \begin{cases} q + Q\Gamma_{k,q} - k, & \text{if } q > \tau_i\,, \\ Q + Q\Gamma_{k,Q} - k, & \text{if } q \leq \tau_i\,. \end{cases}$$

Checking the feasibility of a route r : Strategy II

Monte Carlo sampling and statistical inference.

(the approach becomes probabilistic because there is no guarantee that the optimal solution will always be returned:
Type I: Inferring that r is infeasible when it is feasible; or
Type II: Inferring that r is feasible when it is infeasible.)

Checking the feasibility of a route r : Strategy III

Deriving the distribution of T_r to compute $\mathbb{P}[T_r > D]$:

function ISFEASIBLE(r) ▷ returns TRUE iff r is feasible

$p^{r,1}_{L,P}(Q + Q\Gamma_{k,Q} - k, c_{0,1} + \Gamma_{k,Q}(c_{1,0} + c_{0,1})) = p^k_1$

for $j = 2$ **to** n_r **do**

$p^{r,j}_{L,P}(\cdot,\cdot) \leftarrow \mathbf{0}$ ▷ initialize with zeros

for all l, p **such that** $p^{r,j-1}_{L,P}(l,p) > 0$ **do**

for all k **such that** $p^k_j > 0$ **do**

if $l > \tau_{j-1}$ **then** ▷ visiting j directly

$p^{r,j}_{L,P}(l + Q\Gamma_{k,l} - k, p + t_{s_{j-1}s_j} + \Gamma_{k,l}(c_{j,0} + c_{0,j})) += p^{r,j-1}_{L,P}(l,p)p^k_j$

else ▷ restocking before visiting j

$p^{r,j}_{L,P}(Q + Q\Gamma_{k,Q} - k, p + c_{j-1,0} + c_{0,j} + \Gamma_{k,Q}(c_{j,0} + c_{0,j}) += p^{r,j-1}_{L,P}(l,p)p^k_j$

end if

end for

end for

end for

return $\sum_{l=0}^{Q} \sum_{p \le \overline{T} - c_{n_r,0}} p^{r,n_r}_{L,P}(l,p) \ge \alpha$

end function

Computational Results with $\alpha = 0.95$

	MRV-16		B&P				
Instance	Best	#	Best	#	$\#_{>Q}$	Gap	Time(h)
A-n32-k5	866.77	5	866.45	5	0		5.00
A-n33-k5	735.00	6	732.82	5	1		6.74
A-n33-k6	793.90	6	792.54	5	3		10.64
A-n34-k5	839.01	6	838.95	6	0		0.44
A-n37-k5	713.99	5	710.78	4	1	1.57%	t.l.
A-n38-k5	777.59	6	777.55	6	0		29.58
A-n39-k6	889.40	6	887.04	5	1	0.81%	t.l.
A-n48-k7	1210.79	7	1212.32	7	1	1.60%	t.l.
P-n19-k2	233.36	3	227.87	2	1		0.02
P-n20-k2	240.84	3	240.80	3	0		0.12
P-n21-k2	234.00	3	233.98	3	0		0.09
P-n22-k2	242.19	3	242.18	3	0		0.43
P-n22-k8	715.81	10	591.56	7	1		0.17
P-n23-k8	634.46	10	620.45	7	3		0.14
P-n40-k5	488.50	5	487.61	5	0		0.89
P-n45-k5	539.66	6	539.65	6	0	0.09%	t.l.
P-n50-k8	680.42	9	680.34	9	0	0.24%	t.l.
P-n50-k10	772.25	11	771.49	10	1	0.11%	t.l.
P-n51-k10	833.42	11	832.40	10	1	0.49%	t.l.
P-n55-k10	759.36	11	754.12	10	1	0.55%	t.l.
P-n55-k15	1086.44	17	1075.18	14	4	0.38%	t.l.
P-n60-k10	811.44	11	809.42	10	2	0.17%	t.l.
P-n60-k15	1110.72	17	1101.41	15	3	0.61%	t.l.

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Questions????????????

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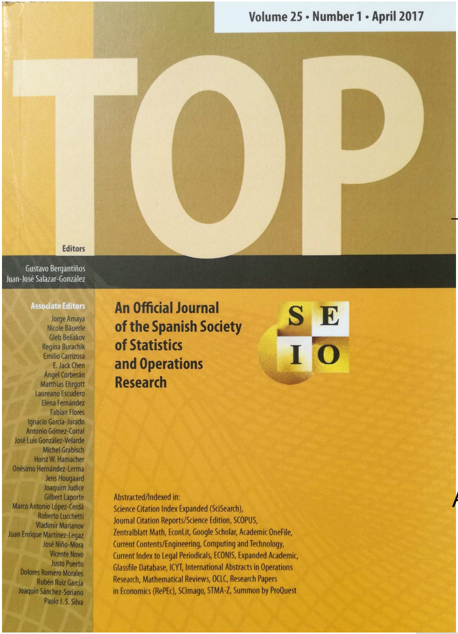

Year	factor	articles	position
2012	0.843	40	44/79
2013	0.766	25	54/79
2014	0.831	57	61/81
2015	0.927	32	55/82
2016	0.899	32	64/83
2017	**1.094**	25	60/83

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