A multi-period bilevel approach for stochastic equilibrium in network expansion planning under uncertainty

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Outline / Summary

- Stochastic Optimization. Some elements and Network Expansion Planning (NEP).
- Primal-dual model: Single-level equivalent multi-period stochastic equilibrium-based optimization model SE-NEP.
- Pilot case: An extension of the classical static Bilevel Tall Assignment Problem (TAP).
- Matheuristic Nested Stochastic Decomposition NSD, very brief presentation.
- Computational experience.
  The large-sized character of the instances is motivated by the intrinsic dimensions of the mixed 0-1 bilinear problem as well as the cardinality of the stochastic network.
- Some conclusions

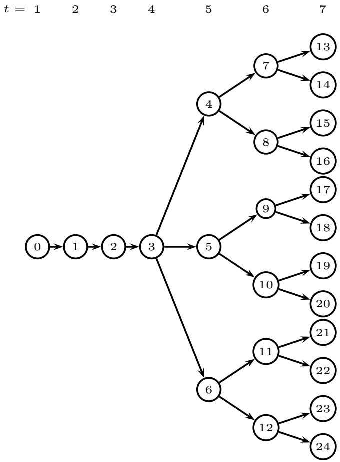
Stochastic Optimization. Some elements

A **period** of a given time horizon is a set of consecutive time units where the realization of the uncertain parameters takes place.

A **scenario** is a realization of the uncertain parameters along the periods of a given time horizon.

A **scenario group** for a given period is the set of scenarios with the same realization of the uncertain parameters up to the period:
Non-anticipativity principle should be satisfied.

A scenario group has one-to-one correspondence with a node in the scenario tree.



$\Omega = \Omega^0 = \{13, 14, \ldots, 24\}$; $\Omega^4 = \{13, 14, 15, 16\}$
$\mathcal{N} = \{0, \ldots, 24\}$; $\mathcal{N}_6 = \{7, 8, 9, 10, 11, 12\}$, $t^6 = 5$
$\mathcal{A}^{11} = \{0, 1, 2, 3, 6, 11\}$, $\sigma^{11} = 6$, $S^6 = \{11, 12, 21, 22, 23, 24\}$

Figura: Multi-period scenario tree

Scenario tree notation

$\mathcal{T}$, set of the time periods along the horizon, $T = |\mathcal{T}|$.

Ω, set of scenarios.

$\mathcal{N}$, set of nodes in the tree.

$\mathcal{A}^n$, set including node n and its ancestor nodes, for $n \in \mathcal{N}$.

$\mathcal{S}_1^n$, set of immediate successor nodes to node n, for $n \in \mathcal{N}$.

Ω^n, set of scenarios with one-to-one correspondence with node n in the scenario tree.

σ^n, immediate ancestor node of node n, for $n \in \mathcal{N}$.

t^n, period to which node n belong to, for $n \in \mathcal{N}$.

w^ω, weight assigned to scenario $\omega \in \Omega$.

$w^n = \sum_{\omega \in \Omega^n} w^\omega$, weight of node $n \in \mathcal{N}$.

Multiperiod Mixed 0-1 DEM for Network Expansion Planning (NEP)

$$\begin{aligned} z = max \sum_{n \in \mathcal{N}} w^n(a^n x^n + b^n y^n) & \\ \text{s.t. } x^n \in \{0,1\}^{nx(n)},\ x^{\sigma^n} \leq x^n & \quad \forall n \in \mathcal{N} \\ A^n x^n + B^n y^n = h^n & \quad \forall n \in \mathcal{N} \\ y^n \in Y^n & \quad \forall n \in \mathcal{N}, \end{aligned}$$

where x^n is a vector included by the so-named 0-1 step vars, and Y^n is a feasible set of mixed 0-1 vectors.

Let $\hat{x}^n$, $\hat{y}^n$: Value vectors of the vectors x^n, y^n, resp., in a (full or partial) solution, if any, for $n \in \mathcal{N}$.

Stochastic Equilibrium-based NEP optimization. TAP: Sets

$\tilde{\mathcal{N}}$, nodes in the network.

$\mathcal{K}$, commodities to transport.

$\tilde{A}_1$, potential directed links in the network nodes by its own means.

$\tilde{A}_2$, directed links in the network nodes by alternative means.

$\tilde{A} = \tilde{A}_1 \cup \tilde{A}_2$, links in the network.

$\Gamma_+^{\tilde{n}} \subset \tilde{A}$, links from node $\tilde{n}$ in the network, for $\tilde{n} \in \tilde{\mathcal{N}}$.

$\Gamma_-^{\tilde{n}} \subset \tilde{A}$, links to node $\tilde{n}$ in the network, for $\tilde{n} \in \tilde{\mathcal{N}}$.

Stochastic Equilibrium-based NEP optimization. TAP: Deterministic parameters

$\overline{\theta}_a$, upper bound on the tariff that is allowed for transporting the commodities through network link a at any period, for $a \in \tilde{A}_1$.

C_a, maintenance cost of network link a, provided that it is available, for $a \in \tilde{A}_1$.

o_k, d_k, origin and destination nodes for commodity k transportation, resp., through the network nodes set $\tilde{\mathcal{N}}$, for $k \in \mathcal{K}$.

Stochastic Equilibrium-based NEP optimization. TAP: Stochastic parameters for scenario tree node $n \in \mathcal{N}$

U_a^n, investment cost for building network link a, for $a \in \tilde{A}_1$.

$\overline{U}^n$, available budget for network expansion planning.

$V_{a,k}^n$, commodity k volume, for $k \in \mathcal{K}$, to be transported by using either network link a for $a \in \tilde{A}_1$ and alternative link a for $a \in \tilde{A}_2$.

$B_{a,k}^n$, commodity k transport unit cost through non-network link a, for $a \in \tilde{A}_2$, $k \in \mathcal{K}$.

Stochastic Equilibrium-based NEP optimization. TAP: Variables x_a^n in the upper-level model for scenario node n, for $n \in \mathcal{N}$

x_a^n, 0-1 step variable whose value 1 means that link a is part of the network **by** node n (and, then, by period t^n) and otherwise, 0, for $a \in \tilde{A}_1$.

$$x_a^n \in \{0,1\}, \;\; x_a^{\sigma^n} \leq x_a^n$$

step variables help to tightening the model in a stronger manner than their counterparts, so-named *impulse* ones.

$x_a^n = 1$: link a is made available *by* scenario node n, i.e., either at the node or at any of its ancestors in the scenario tree, such that:
for the former case, $x_a^n - x_a^{\sigma^n} = 1$ and,
for the latter one, $x_a^n = x_a^{\sigma^n} = 1$.

Stochastic Equilibrium-based NEP optimization. TAP: Variables θ_a^n in the upper-level model for scenario node n, for $n \in \mathcal{N}$

θ_a^n, unit toll price (i.e., tariff) for commodity transport through network link a, for $a \in \tilde{A}_1$.

$$0 \leq \theta_a^n \leq \overline{\theta}_a$$

It is not active in case that link a is not available, i.e., $x_a^n = 0$.

Variables in (each) dupla (commodity k, scenario node n)-based lower-level model, for $k \in \mathcal{K}$, $n \in \mathcal{N}$

The variables in (each) lower-level model are related to the flow of the commodities k through the nodes of the network, by using either the network available links or other means depending on the transport cost of other alternatives at each period of the time horizon.

$y^n_{a,k}$, 0-1 variable whose value 1 means that network link a is used for transporting commodity k in scenario node n and otherwise, 0, for $a \in \tilde{A}_1$.

$$y^n_{a,k} \in \{0,1\},\ y^n_{a,k} \leq x^n_a$$

$z^n_{a,k}$, 0-1 variable whose value 1 means that alternative link a is used for transporting commodity k in scenario node n and otherwise, 0, for $a \in \tilde{A}_2$.

- Goal: Looking for the Stackelberg (1934) equilibrium between the Leader's expected profit and the Follower's expected transport cost.

  That is, the solution of the upper-level model is influenced by the optimization of the (each) lower-level model, and vice versa.

- The maximization of the expected profit in the scenarios of the upper-level model has an additional type of constraint:

  In (each) lower-level model, minimizing the commodity k transport cost in the available network links and the transport cost of the alternative means, **for each scenario node**, for $k \in \mathcal{K}$, $n \in \mathcal{N}$.

- Modeling scheme: A tariff-based network's single model to be solved up to optimality for each scenario node: **Coordinated primal-dual** submodels.

Upper-level model

$$p = max \sum_{n \in \mathcal{N}} w^n$$

$$\Big[\sum_{k \in \mathcal{K}} \sum_{a \in \tilde{A}_1} V^n_{a,k} \theta^n_a y^n_{a,k} - \sum_{a \in \tilde{A}_1} \big(U^n_a (x^n_a - x^{\sigma^n}_a) + C_a x^n_a \big) \Big] \quad (2a)$$

$$\text{s.t. } 0 \leq \theta^n_a \leq \overline{\theta}_a \qquad \forall a \in \tilde{A}_1,\ n \in \mathcal{N} \quad (2b)$$

$$x^n_a \in \{0,1\},\ x^{\sigma^n}_a \leq x^n_a \qquad \forall a \in \tilde{A}_1,\ n \in \mathcal{N} \quad (2c)$$

$$\sum_{a \in \tilde{A}_1} U^n_a (x^n_a - x^{\sigma^n}_a) \leq \overline{U}^n \qquad \forall n \in \mathcal{N} \quad (2d)$$

$$y^n_{a,k} \in \{0,1\},\ y^n_{a,k} \leq x^n_a \qquad \forall a \in \tilde{A}_1,\ k \in \mathcal{K},\ n \in \mathcal{N} \quad (2e)$$

(k, n)-scenario node-based lower-level model, for $k \in \mathcal{K}$, $n \in \mathcal{N}$

$$c_k^n = min \sum_{a \in \tilde{A}_1} V_{a,k}^n \theta_a^n y_{a,k}^n + \sum_{a \in \tilde{A}_2} V_{a,k}^n B_{a,k}^n z_{a,k}^n \tag{3a}$$

$$\text{s.t.} \sum_{a \in \Gamma_+^{\tilde{n}} \cap \tilde{A}_1} y_{a,k}^n + \sum_{a \in \Gamma_+^{\tilde{n}} \cap \tilde{A}_2} z_{a,k}^n - \sum_{a \in \Gamma_-^{\tilde{n}} \cap \tilde{A}_1} y_{a,k}^n - \sum_{a \in \Gamma_-^{\tilde{n}} \cap \tilde{A}_2} z_{a,k}^n$$

$$= \rho_{k\tilde{n}} \qquad \forall k \in \mathcal{K},\ \tilde{n} \in \tilde{\mathcal{N}} \tag{3b}$$

$$0 \leq y_{a,k}^n \leq 1 \qquad \forall a \in \tilde{A}_1 \tag{3c}$$

$$0 \leq z_{a,k}^n \leq 1 \qquad \forall a \in \tilde{A}_2 \tag{3d}$$

where for $\tilde{n} \in \tilde{\mathcal{N}}$: $\rho_{k\tilde{n}} = -1$ for $\tilde{n} = d_o$, $\rho_{k\tilde{n}} = 1$ for $\tilde{n} = d_k$ and otherwise, $\rho_{k\tilde{n}} = 0$.

- It is well-known that there is an optimal solution for model (3) where the y, z-variables take integer values.
- Notice that there is a submodel for each scenario node, at least, as is it is usual in the NEP applications:

  The lower-level reacts to each offered upper-level solution then, in TAP as an illustrative example, there is a reaction to each set of θ-,x-variables.

Single model scheme

- The selection of the type of single model is a subject of current research.
- Alternative scheme: Appending to the upper-level primal

  model (2) and constraint systems of the $|\mathcal{K}| \times |\mathcal{N}|$ lower-level primal models (3),

  either the (k, n)-based KKT constraint system of the (each) (k, n)-lower-level primal models,

  or its (k, n)-based dual models jointly with the equating of the n-based primal and dual objective functions,
  for $n \in \mathcal{N}$.

- However, the primal-dual approach is chosen in this work, since provisional results on the KKT approach required unaffordable computing effort for instances under consideration.
- That big effort, probably, was due to the big M-parameters that are required for the linearly representation of the quadratic complementary slackness conditions for each scenario node.
- Let $\delta^{n}_{\tilde{n}',k}$ be the free dual variable of the conservation flow constraint (3b) of the primal-dual model below, for the triplet given by network node $\tilde{n}'$, commodity k and scenario node n.

Single deterministic equivalent model to the multi-period stochastic equilibrium-based primal-dual optimization model for network expansion planning

Upper level primal model

$$\hat{p} = max \sum_{n \in \mathcal{N}} w^n$$

$$[\sum_{k \in \mathcal{K}} \sum_{a \in \tilde{A}_1} V^n_{a,k} \theta^n_a y^n_{a,k} - \sum_{a \in \tilde{A}_1} \left(U^n_a (x^n_a - x^{\sigma^n}_a) + C_a x^n_a \right)] \quad (4a)$$

$$\text{s.t. } 0 \leq \theta^n_a \leq \overline{\theta}_a \qquad \forall a \in \tilde{A}_1, \, n \in \mathcal{N} \quad (4b)$$

$$x^n_a \in \{0, 1\}, \; x^{\sigma^n}_a \leq x^n_a \qquad \forall a \in \tilde{A}_1, \, n \in \mathcal{N} \quad (4c)$$

$$\sum_{a \in \tilde{A}_1} U^n_a (x^n_a - x^{\sigma^n}_a) \leq \overline{U}^n \qquad \forall n \in \mathcal{N} \quad (4d)$$

$$y^n_{a,k} \in \{0, 1\}, \; y^n_{a,k} \leq x^n_a \qquad \forall a \in \tilde{A}_1, \, k \in \mathcal{K}, \, n \in \mathcal{N} \quad (4e)$$

s.t. primal and dual n-based lower level models, for $n \in \mathcal{N}$

Primal model constraints:

$$\sum_{a \in \Gamma_+^{\tilde{n}} \cap \tilde{A}_1} y_{a,k}^n + \sum_{a \in \Gamma_+^{\tilde{n}} \cap \tilde{A}_2} z_{a,k}^n - \sum_{a \in \Gamma_-^{\tilde{n}} \cap \tilde{A}_1} y_{a,k}^n - \sum_{a \in \Gamma_-^{\tilde{n}} \cap \tilde{A}_2} z_{a,k}^n = \rho_{k\tilde{n}} \quad \forall \tilde{n} \in \tilde{\mathcal{N}},\ k \in \mathcal{K} \tag{4f}$$

$$0 \leq z_{a,k}^n \leq 1 \quad \forall a \in \tilde{A}_2,\ k \in \mathcal{K} \tag{4g}$$

Dual model constraints:

$$V_{a,k}^n \theta_a^n + \delta_{\tilde{n},k}^n - \delta_{\tilde{n}',k}^n \geq 0 \quad \forall a = (\tilde{n}, \tilde{n}') \in \tilde{A}_1,\ k \in \mathcal{K} \tag{4h}$$

$$V_{a,k}^n B_{a,k}^n + \delta_{\tilde{n},k}^n - \delta_{\tilde{n}',k}^n \geq 0 \quad \forall a = (\tilde{n}, \tilde{n}') \in \tilde{A}_2,\ k \in \mathcal{K} \tag{4i}$$

Equating n-based primal and dual objective functions:

$$\sum_{k\in\mathcal{K}}\Big(\sum_{a\in\tilde{A}_1} V^n_{a,k}\theta^n_a y^n_{a,k} + \sum_{a\in\tilde{A}_2} V^n_{a,k}B^n_{a,k}z^n_{a,k}\Big) = \sum_{k\in\mathcal{K}}\big(\delta^n_{d_k,k} - \delta^n_{o_k,k}\big) \tag{4j}$$

NSD matheuristic, Nested Stochastic Decomposition

- NSD starts by grouping consecutive time periods into modeler-driven so-named stages, creating thus a collection of subtrees/subproblems.
- The subproblems are not independent but each one is linked by the 0-1 step vars to the immediate ancestor subproblem.
- NSD solves the subproblems at each iteration, by executing first a front-to-back scheme and after a back-to-from scheme.

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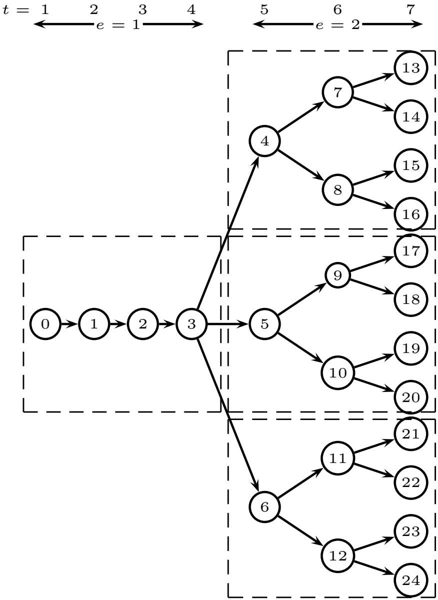

$\mathcal{G}_2 = \{4, \ldots, 24\}$

$\mathcal{R}_2 = \{4, 5, 6\}$

$\mathcal{C}^4 = \{4, 7, 8, 13, \ldots, 16\}$

$\mathcal{L}^0 = \{3\}$

Scenario subtrees supporting the NSD subproblems. Sets

$\mathcal{E}$, stages in the time horizon.

$\mathcal{T}_e$, set of periods in stage e, such that $\mathcal{T} = \sum_{e \in \mathcal{E}} \mathcal{T}_e$, $\mathcal{T}_e \cap \mathcal{T}_{e'} = \emptyset : e \neq e'$.

$\mathcal{G}_e \subseteq \mathcal{N}$, set of nodes in stage e, for $e \in \mathcal{E}$.

$\mathcal{R}_e \subseteq \mathcal{G}_e$, root nodes of the subtrees in stage e, for $e \in \mathcal{E}$.

$\mathcal{C}^r \subseteq \mathcal{G}_e$, nodes that belong to the subtree rooted with node r, for $r \in \mathcal{R}_e$, $e \in \mathcal{E}$.

$\mathcal{L}^r \subseteq \mathcal{C}^r$, leaf nodes in subtree related to set $\mathcal{C}^r$, for $r \in \mathcal{R}_e$, $e \in \mathcal{E}$.

Note: The 0-1 step x^ℓ-vars are the state ones, i.e., they are the only vars in a subproblem supported by the subtree rooted with node r that have nonzero elements in *constraints associated with the root nodes in the immediate successor subproblems* to leaf node ℓ, being defined by the node set $\bigcup_{r' \in \mathcal{S}_1^\ell} \mathcal{C}^{r'}$, for $\ell \in \mathcal{L}^r$, $r \in \mathcal{R}_e$, $e \in \mathcal{E} : e < E$.

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NSD matheuristic. EFV curves

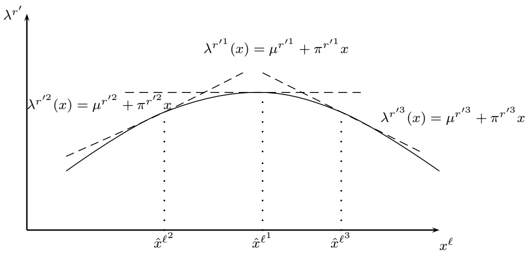
- Let the so-named Expected Future Value (EFV) curve of the subproblem supported by the subtree given by nodeset $\mathcal{C}^{r'}$, whose root node r' is an immediate successor of leaf node ℓ, for $r' \in \mathcal{S}_1^{\ell}$, $\ell \in \mathcal{L}^r$, $r \in \mathcal{R}_e$, $e \in \mathcal{E} : e < E$, where remember $\mathcal{L}^r \subseteq \mathcal{C}^r$.

  EFV curves estimate the impact of the state vars (decisions) of the leaf nodes.

EFV curve $\lambda'^{r'}(x^\ell)$, for $r' \in \mathcal{S}_1^\ell$, $\ell \in \mathcal{L}^r$, $r \in \mathcal{R}_e$, $e \in \mathcal{E} : e < E$

- $\lambda'^{r'}(x^\ell)$, (assumed convex) curve of the expected future objective function value (EFV) for the set of scenarios, say $\Omega^{r'}$, in the full subtree rooted with node r', for $r' \in \mathcal{S}_1^\ell$.

  In general, $\lambda'^{r'}(x^\ell)$ is difficult to compute.
- NSD approach approximates it with the piecewise linear convex function to be referred to as EFV function $\lambda^{r'}$.



For easing the exposition, let a set of variables in the original model (4) be notated as vector Z^n, for $n \in \mathcal{N}$. It can be expressed

$$Z^n \equiv \Big(\big(\theta_a^g \, \forall a \in \tilde{A}_1; \\ (y_{a,k}^g \, \forall a \in \tilde{A}_1, \, k \in \mathcal{K}; \; z_{a,k}^g \, \forall a \in \tilde{A}_2, \, k \in \mathcal{K}); \\ \delta_{\tilde{n},k}^g \, \forall \tilde{n} \in \tilde{\mathcal{N}}, \, k \in K; \, \forall g \in \mathcal{A}^n; \big) \\ \big(x_a^g \, \forall a \in \tilde{A}_1 \; g \in \mathcal{A}^n \setminus \{n\}\big)\Big) \tag{5}$$

Remember that $(\hat{.})$ denote the current solution of the variables' vector (.).

Approximation subproblem supported by subtree composed of nodeset $\mathcal{C}^r$, for $r \in \mathcal{R}_e$, $e \in \mathcal{E} : e < E$

$$F^r = max \sum_{\ell \in \mathcal{L}^r} w^\ell$$

$$[\sum_{n \in \mathcal{A}^\ell} [\sum_{k \in \mathcal{K}} \sum_{a \in \tilde{A}_1} V^n_{a,k} \theta^n_a y^n_{a,k} - \sum_{a \in \tilde{A}_1} (U_a(x^n_a - x^{\sigma^n}_a) + C_a x^n_a)]$$

$$+ \sum_{r' \in \mathcal{S}^\ell_1} \frac{w^{r'}}{w^\ell} \lambda^{r'}] \quad (6a)$$

s.t. cons system in model (4) where $\forall n \in \mathcal{N}$ is $\forall n \in \mathcal{C}^r$ (6b)

$$x^n \in \{0,1\}^{nx}, \; x^{\sigma^n} \leq x^n \quad \forall n \in \mathcal{C}^r \quad (6c)$$

$$Z^{\sigma^r} = \hat{Z}^{\sigma^{r^{\hat{q}}}} \quad (6d)$$

$$x^{\sigma^r} - \hat{x}^{\sigma^{r^{\hat{q}}}} = 0 \quad (6e)$$

$$\lambda^{r'} \leq \mu^{r'q} + \pi^{r'q} x^\ell \quad \forall q \in \mathcal{Q}^\ell, \; r' \in \mathcal{S}^\ell_1, \; \ell \in \mathcal{L}^r. \quad (6f)$$

On obtaining vector $(\mu^{r\hat{q}}, \pi^{r\hat{q}})$

It is worth to pointing out that some state-of-the-art optimizers (as CPLEX and others) do not provide the duals of the constraints for fixings vars in MIP models.
So, some alternatives:

- Benders cut, *B* (Benders NM'62).
- Lagrangean cut, *L*
  (Zou, Ahmed & Sun OptimOnline'16, MPB'18).
- Strengthened Benders cut, *SB*
  (Zou, Ahmed & Sun OptimOnline'16, MPB'18).
- Heuristic Benders cut, *HB*

- A mixture of *SB* cuts for later sages and *L* cuts for earlier stages, in particular, stage $e = 1$. Work-in-Progress.

Computing HW/SW

- A processor Intel(R) Xeon(R) CPU E5-2650 v3 @ 2.30GHz, 20 cores, RAM 62 GiB.
- A C++ experimental code
- CPLEX v12.8, using between 15 and 20 cores.
- NSD parameters: $\epsilon_1 = 0{,}0001$, $\epsilon_2 = 0{,}0001$,
  iters limit $\overline{q} = 100$,
  Lagrangean iters limit $\tilde{q} = 4$.
- Given the difficulty of the instances, $E = T$ and, so, $|\mathcal{T}_e| = 1$, $\mathcal{T}_e = \{t^{r'}\}$, $|\mathcal{C}^{r'}| = 1$, for $r' \in \mathcal{R}_e$, $e \in \mathcal{E}$,

  $t^{r'}$, period which scenario node r' belongs to.

Cuadro: Problem and model (4) dimensions. Testbed 1

Ins	$\lvert\tilde{N}\rvert$	$\lvert\mathcal{A}_1\rvert$	$\lvert\mathcal{A}_2\rvert$	$\lvert\mathcal{T}\rvert$	*b*	$\lvert\mathcal{N}\rvert$	$\lvert\Omega\rvert$	*m*	*nc*	*n*01
i1	12	4	13	2	2	3	2	1950	474	822
i2	12	4	13	2	3	4	3	2600	632	1096
i3	15	8	18	2	2	3	2	2946	675	1212
i4	12	4	13	2	4	5	4	3250	790	1370
i5	15	8	18	2	3	4	3	3928	900	1616
i6	12	4	13	2	5	6	5	3900	948	1644
i7	12	4	13	3	2	7	4	4550	1106	1918
i8	15	8	18	2	4	5	4	4910	1125	2020
i9	16	6	18	2	2	3	2	4797	1125	2034
i10	15	8	18	2	5	6	5	5892	1350	2424
i11	16	6	18	2	3	4	3	6396	1500	2712
i12	15	8	18	3	2	7	4	6874	1575	2828
i13	20	12	25	2	2	3	2	7329	1623	3012
i14	16	6	18	2	4	5	4	7995	1875	3390
i15	12	4	13	3	3	13	9	8450	2054	3562
i16	20	12	25	2	3	4	3	9772	2164	4016

Cuadro: Problem and model (4) dimensions. Testbed 2

Ins	$\|\tilde{N}\|$	$\|\mathcal{A}_1\|$	$\|\mathcal{A}_2\|$	$\|\mathcal{T}\|$	*b*	$\|\mathcal{N}\|$	$\|\Omega\|$	*m*	*nc*	*n*01
i17	16	6	18	2	5	6	5	9594	2250	4068
i18	12	4	13	4	2	15	8	9750	2370	4110
i19	16	6	18	3	2	7	4	11193	2625	4746
i20	20	12	25	2	4	5	4	12215	2705	5020
i21	15	8	18	3	3	13	9	12766	2925	5252
i22	12	4	13	3	4	21	16	13650	3318	5754
i23	20	12	25	2	5	6	5	14658	3246	6024
i24	15	8	18	4	2	15	8	14730	3375	6060
i25	20	12	25	3	2	7	4	17101	3787	7028
i26	15	8	18	3	4	21	16	20622	4725	8484
i27	12	4	13	5	2	31	16	20150	4898	8494
i28	12	4	13	3	5	31	25	20150	4898	8494
i29	16	6	18	3	3	13	9	20787	4875	8814
i30	16	6	18	4	2	15	8	23985	5625	10170
i31	12	4	13	4	3	40	27	26000	6320	10960
i32	15	8	18	5	2	31	16	30442	6975	12524

Cuadro: Problem and model (4) dimensions. Testbed 3

Ins	$\|\tilde{N}\|$	$\|\mathcal{A}_1\|$	$\|\mathcal{A}_2\|$	$\|\mathcal{T}\|$	*b*	$\|\mathcal{N}\|$	$\|\Omega\|$	*m*	*nc*	*n*01
i33	15	8	18	3	5	31	25	30442	6975	12524
i34	20	12	25	3	3	13	9	31759	7033	13052
i35	16	6	18	3	4	21	16	33579	7875	14238
i36	20	12	25	4	2	15	8	36645	8115	15060
i37	15	8	18	4	3	40	27	39280	9000	16160
i38	12	4	13	6	2	63	32	40950	9954	17262
i39	16	6	18	5	2	31	16	49569	11625	21018
i40	16	6	18	3	5	31	25	49569	11625	21018
i41	20	12	25	3	4	21	16	51303	11361	21084
i42	12	4	13	4	4	85	64	55250	13430	23290
i43	15	8	18	6	2	63	32	61866	14175	25452
i44	16	6	18	4	3	40	27	63960	15000	27120
i45	20	12	25	5	2	31	16	75733	16771	31124
i46	20	12	25	3	5	31	25	75733	16771	31124
i47	12	4	13	5	3	121	81	78650	19118	33154
i48	15	8	18	4	4	85	64	83470	19125	34340

Cuadro: Problem and model (4) dimensions. Testbed 4

Ins	$\|\tilde{N}\|$	$\|\mathcal{A}_1\|$	$\|\mathcal{A}_2\|$	$\|\mathcal{T}\|$	b	$\|\mathcal{N}\|$	$\|\Omega\|$	m	nc	$n01$
i49	12	4	13	7	2	127	64	82550	20066	34798
i50	20	12	25	4	3	40	27	97720	21640	40160
i51	16	6	18	6	2	63	32	100737	23625	42714
i52	12	4	13	4	5	156	125	101400	24648	42744
i53	15	8	18	5	3	121	81	118822	27225	48884
i54	15	8	18	7	2	127	64	124714	28575	51308
i55	16	6	18	4	4	85	64	135915	31875	57630
i56	15	8	18	4	5	156	125	153192	35100	63024
i57	20	12	25	6	2	63	32	153909	34083	63252
i58	16	6	18	5	3	121	81	193479	45375	82038
i59	20	12	25	4	4	85	64	207655	45985	85340
i60	16	6	18	7	2	127	64	203073	47625	86106
i61	16	6	18	4	5	156	125	249444	58500	105768
i62	20	12	25	5	3	121	81	295603	65461	121484
i63	20	12	25	7	2	127	64	310261	68707	127508
i64	20	12	25	4	5	156	125	381108	84396	156624

instances i1– i32 instances i33 – i64

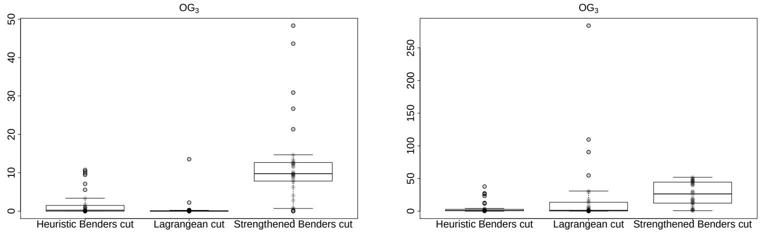


Figura: NSD optimality gap OG_3 % in the cut options $L4$ and SB. NSD gap estimation $\hat{OG}_3$ in HB

- Except for the outliers, the optimality gap OG_3 is smaller in $L4$ than SB, what is not a surprise.
- $\hat{OG}_3$ for HB: A good estimation of OG_3

Cuadro: Average results for small-middle instances i1-i32

Cut option	t_3	$\hat{q}$	GR_1
Lagrangean	1572	10.4	1.00
Strengthened Benders	425	7.9	1.01
Heuristic Benders	328	8.2	1.04

Cuadro: Average results for large instances i33-i64

Cut option	t_3	$\hat{q}$	GR_1
Lagrangean	54106	22.4	1.09
Strengthened Benders	11313	9.5	1.17
Heuristic Benders	11722	18.8	1.24

Conclusions. SE-NEP, Stochastic-Equilibrium in multi-period Network Expansion Planning

- E-NEP, Multi-period planning under uncertainty
- Difficult problem: Uncertainty, mixed 0-1 bilinear NEP, 1 upper-level and as many lower-levels as scenario nodes.
- Single primal-dual modeling scheme of choice.
- Strong *step 0-1 vars* type (i.e., *by* occurring in scenario node n): $x^n \in \{0,1\}$ where $x^{\sigma^n} \leq x^n$ for $n \in \mathcal{N}$ versus *impulse 0-1* ones (i.e., *at* occurring in scenario node n): $x^n \in \{0,1\}$ where $\sum_{n \in \mathcal{A}^\omega} x^n \leq 1$ for $\omega \in \Omega$.
- Decomposition methodology, *a must* for problem solving in large-sized instances,
  due to dimensions of problem network and scenario tree.

Conclusions. NSD, Nested Stochastic Decomposition

- For single primal-dual model, NSD proposal: Good results versus CPLEX, a M01LP/M01QP state-of-the-art solver.
- NSD: A good tool for helping to solve large-sized very difficult problems.
- Better for state variables *only in consecutive two periods* .

  NSD performance improving: *only step 0-1 vars*

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