An extension of the p-center problem considering stratified demand

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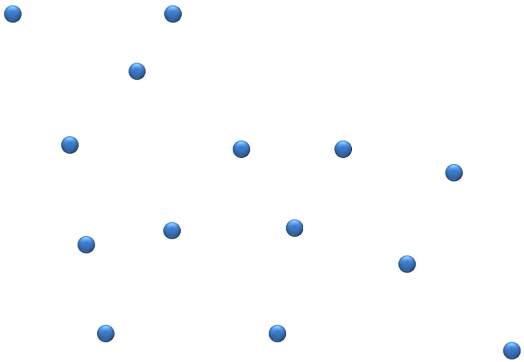
Outline

The discrete p-center problem



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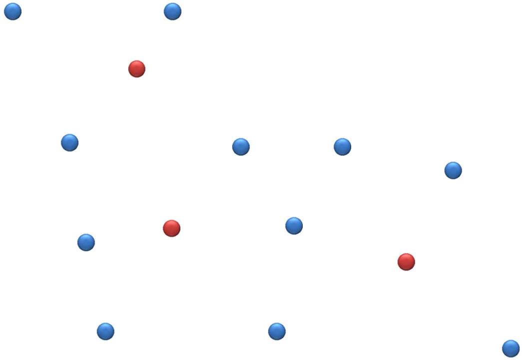

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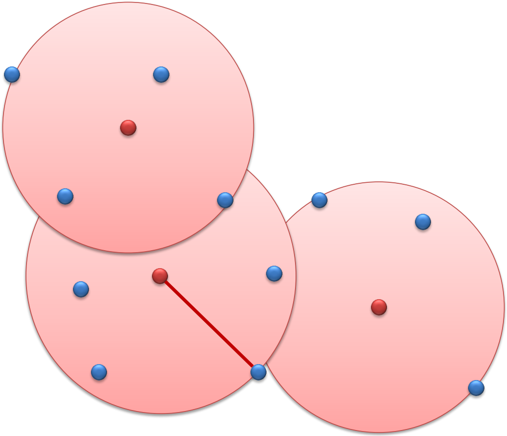
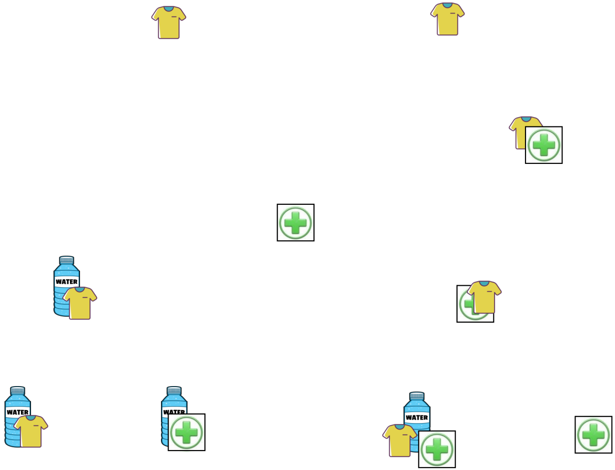


The stratified p-center problem (SpCP): Introduction



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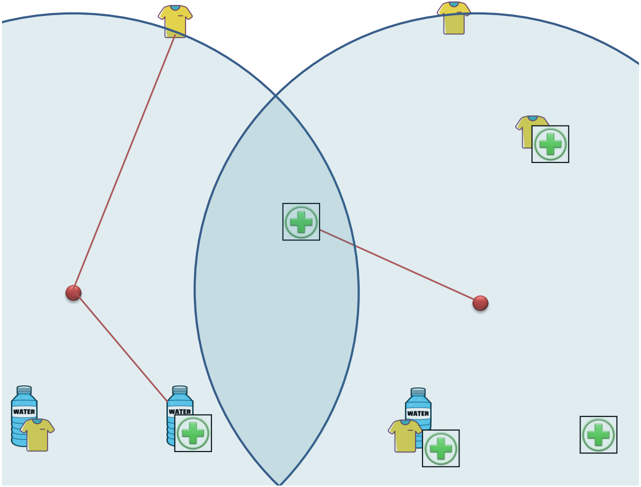

The SpCP: Notation and description

- $J = \{1, \ldots, n\}$ potential locations for centers.
- $I = \{1, \ldots, m\}$ demand sites $(J = I)$.
- $\mathcal{S}$ set of strata.
  - J^s is the set of sites where stratum s is present, for $s \in \mathcal{S}$.
  - w_s weight associated with each stratum $s \in \mathcal{S}$.

$$\min_{\substack{P \subseteq J \\ |P| = p}} \sum_{s \in \mathcal{S}} w_s d(J^s, P),$$

$$\left(\text{where } d(J^s, P) = \max_{i \in J^s} \min_{j \in P} d_{ij} \right)$$

Formulations for the SpCP

- Formulation inspired by the pCP formulation in Daskin (1995) (F1)

M. Daskin.
Network and Discrete Location: Models, Algorithms, and Applications.
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Computers and Operations Research, 40(12):2991-2999, 2013.

- Formulation using site- and stratum-covering variables. (F3)

A. Marín, S. Nickel, J. Puerto, and S. Velten.
A flexible model and efficient solution strategies for discrete location problems.
Discrete Applied Mathematics, 157(5):1128–1145, 2009.

F1 for SpCP inspired by Daskin (1995)

$$x_{ij} = \begin{cases} 1, & \text{if site } i \to j \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i,j \in J.$$

$$\theta^s = \text{largest allocation distance for the sites in } s,\ s \in \mathcal{S}$$

$$\begin{aligned}
\min \ & \sum_{s\in\mathcal{S}} w_s \theta^s \\
\text{s.t.} \ & \sum_{j\in J} x_{jj} = p, \\
& \sum_{j\in J} x_{ij} = 1, && i \in J, \\
& x_{ij} \leqslant x_{jj}, && i,j \in J, \\
& \theta^s \geqslant \sum_{j\in J} d_{ij} x_{ij}, && s \in \mathcal{S}, i \in J^s, \\
& x_{ij} \in \{0,1\}, && i,j \in J, \\
& \theta^s \geqslant 0, && s \in \mathcal{S}.
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1. Sorted distances within a stratum:

$$0 = d^s_{(1)} < d^s_{(2)} < \ldots < d^s_{(G^s)},$$

2. The sorted distances from a site $i \in J$ to the remaining sites are

$$0 = d_{i(1)} < d_{i(2)} < \ldots < d_{i(G_i)}.$$

F2 for SpCP inspired by Calik and Tansel (2013)

$$\tilde{u}_{sk} = \begin{cases} 1, & \text{if } d^s_{(k)} \text{ is the largest} \\ & \text{allocation distance in } J^s, \\ 0, & \text{otherwise,} \end{cases}$$
$$s \in \mathcal{S}, k = 1, \ldots, G^s.$$

$$y_j = \begin{cases} 1, & \text{if a center is placed at } j, \\ 0, & \text{otherwise,} \end{cases}$$
$$\text{for } j \in J.$$

$$\begin{aligned}
\min \ & \sum_{s \in \mathcal{S}} \sum_{k=1}^{G^s} w_s d^s_{(k)} \tilde{u}_{sk} \\
\text{s.t.} \ & \sum_{j \in J} y_j = p, \\
& \sum_{k=1}^{G^s} \tilde{u}_{sk} = 1, && s \in \mathcal{S}, \\
& \sum_{k'=1}^{k-1} \tilde{u}_{sk'} \leqslant \sum_{\substack{j \in J \\ d_{ij} < d^s_{(k)}}} y_j, s \in \mathcal{S}, i \in J^s, \\
& && k = 2, \ldots, G^s, \\
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$$u_{sk} = \begin{cases} 1, & \text{if the largest distance} \\ & \text{in } J^s \text{ is at least } d^s_{(k)} \\ 0, & \text{otherwise,} \end{cases}$$

for $s \in \mathcal{S},\ k = 2, \dots, G^s$.

$$z_{ir} = \begin{cases} 1, & \text{if the allocation distance} \\ & \text{of client } i \text{ is at least } d_{i(r)}, \\ 0, & \text{otherwise,} \end{cases}$$

$i \in J, r = 2, \dots, G_i$.

$$\min \sum_{s \in \mathcal{S}} \sum_{k=2}^{G^s} w_s(d^s_{(k)} - d^s_{(k-1)}) u_{sk}$$

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$$\begin{aligned} u_{sk} &\geqslant z_{i,\bar{l}^s_{ik}}, && s \in \mathcal{S}, i \in J^s, \\ & && k = 2, \dots, l^s_{iG_i} : \bar{l}^s_{ik} > 0, \\ u_{s,k-1} &\geqslant u_{sk}, && s \in \mathcal{S}, k = 3, \dots, G^s, \\ u_{sk} &\in \{0,1\}, && s \in \mathcal{S}, k = 2, \dots, G^s, \\ z_{ir} &\in \{0,1\}, && i \in J, r = 2, \dots, G_i. \end{aligned} \quad (1)$$

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\end{aligned}$$

F3:Alternatives

Proposition

Constraints (1) can be replaced by any of the next families of constraints.

$$u_{sk} \geqslant z_{i,l'^{s}_{ik}}, \qquad s \in \mathcal{S}, i \in J^s, k = 2, \dots, l^s_{iG_i}. \tag{2}$$

$$n_{sk} u_{sk} \geqslant \sum_{\substack{i \in J^s \\ \bar{l}^s_{ik} \neq 0}} z_{i\bar{l}^s_{ik}} \quad s \in \mathcal{S}, k = 2, \dots, G^s. \tag{3}$$

$$|J^s| u_{sk} \geqslant \sum_{\substack{i \in J^s \\ l'^{s}_{ik} \leqslant G_i}} z_{i,l'^{s}_{ik}}, \quad s \in \mathcal{S}, k = 2, \dots, G^s. \tag{4}$$

$$\sum_{\substack{s \in \mathcal{S}: \xi^s_i = 1 \\ l^s_{ir} \geqslant 2}} u_{s,l^s_{ir}} \geqslant \left(\sum_{s \in \mathcal{S}} \xi^s_i \right) z_{ir} \quad i \in J, r = 2, \dots, G_i. \tag{5}$$

F3:Alternatives

Proposition

Constraints (1) can be replaced by any of the next families of constraints.

$$u_{sk} \geqslant z_{i,l'^s_{ik}}, \qquad s \in \mathcal{S}, i \in J^s, k = 2, \ldots, l^s_{iG_i}. \tag{2}$$

$$n_{sk} u_{sk} \geqslant \sum_{\substack{i \in J^s \\ \bar{l}^s_{ik} \neq 0}} z_{i\bar{l}^s_{ik}} \quad s \in \mathcal{S}, k = 2, \ldots, G^s. \tag{3}$$

$$|J^s| u_{sk} \geqslant \sum_{\substack{i \in J^s \\ l'^s_{ik} \leqslant G_i}} z_{i,l'^s_{ik}}, \quad s \in \mathcal{S}, k = 2, \ldots, G^s. \tag{4}$$

$$\sum_{\substack{s \in \mathcal{S}: \xi^s_i = 1 \\ l^s_{ir} \geqslant 2}} u_{s,l^s_{ir}} \geqslant \left(\sum_{s \in \mathcal{S}} \xi^s_i \right) z_{ir} \quad i \in J, r = 2, \ldots, G_i. \tag{5}$$

Reducing the number of covering variables

Proposition

The largest distance associated with s is at least $v(pCP_s)$ in the opt. sol. of the SpCP.

- For each stratum s:
  - Obtain a lower bound on the largest distance in s (LB_s): pCP_s Binary Algorithm (Calik and Tansel, 2013).
  - Define u_{sk} variables for $k \in \{h : 2 \leqslant h \leqslant G^s \text{ and } d^s_{(h)} > LB_s\}$.

Reducing the number of covering variables

$s:1$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136
$s:2$	1	28	30	31	46	47	59	60	75	76	77	105	106	136	
$s:3$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136
$s:4$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136
$s:5$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136
$s:6$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136
$s:7$	1	28	29	30	46	47	59	60	75	76	77	105	106	136	
$s:8$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136
$s:9$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136
$s:10$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136

Reducing the number of covering variables

$s:1$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136
$s:2$	1	28	30	31	46	47	59	60	75	76	77	105	106	136	
$s:3$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136
$s:4$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136
$s:5$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136
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$s:7$	1	28	29	30	46	47	59	60	75	76	77	105	106	136	
$s:8$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136
$s:9$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136
$s:10$	1	28	29	30	31	46	47	59	60	75	76	77	105	106	136

Reducing the number of covering variables

$s:1$	~~1~~	~~28~~	~~29~~	~~30~~	31	46	47	59	60	75	76	77	105	106	136
$s:2$	~~1~~	~~28~~	~~31~~	~~31~~	46	47	59	60	75	76	77	105	106	136	
$s:3$	~~1~~	~~28~~	~~29~~	~~30~~	~~31~~	46	47	59	60	75	76	77	105	106	136
$s:4$	~~1~~	~~28~~	~~29~~	~~30~~	~~31~~	46	47	59	60	75	76	77	105	106	136
$s:5$	~~1~~	~~28~~	~~29~~	~~30~~	~~31~~	46	47	59	60	75	76	77	105	106	136
$s:6$	~~1~~	~~28~~	~~29~~	~~30~~	~~31~~	46	47	59	60	75	76	77	105	106	136
$s:7$	~~1~~	~~28~~	~~29~~	~~30~~	46	47	59	60	75	76	77	105	106	136	
$s:8$	~~1~~	~~28~~	~~29~~	~~30~~	~~31~~	46	47	59	60	75	76	77	105	106	136
$s:9$	~~1~~	~~28~~	~~29~~	~~30~~	~~31~~	46	47	59	60	75	76	77	105	106	136
$s:10$	~~1~~	~~28~~	~~29~~	~~30~~	~~31~~	46	47	59	60	75	76	77	105	106	136

Valid inequalities for F3:

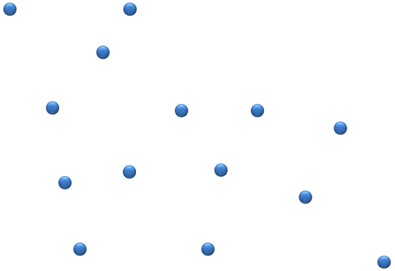
Proposition

Following inequalities are valid for formulation F3.

- $$z_{ir} \leqslant z_{j2} \quad i,j \in J, r = 2, \dots G_i : d_{i(r-1)} = d_{ij}, \tag{6}$$
- $$\sum_{k=2}^{G^{s_1}} (d^{s_1}_{(k)} - d^{s_1}_{(k-1)}) u_{s_1 k} \leqslant \sum_{k=2}^{G^{s_2}} (d^{s_1}_{(k)} - d^{s_1}_{(k-1)}) u_{s_2 k}, \quad s_1, s_2 \in S : J^{s_1} \subseteq J^{s_2}. \tag{7}$$
- $$u_{s_1 k} \leqslant u_{s_2 l}, \quad s_1, s_2 \in S, k = 2, \dots, G^{s_1}, l = 2, \dots, G^{s_2} : J^{s_1} \subseteq J^{s_2}, d^{s_1}_{(k)} = d^{s_2}_{(l)}, \tag{8}$$
- $$\sum_{k=2}^{G^{s_1}} u_{s_1 k} \leqslant \sum_{k=2}^{G^{s_2}} u_{s_2 k}, \quad s_1, s_2 \in S : J^{s_1} \subseteq J^{s_2}. \tag{9}$$
- $$z_{ir} \geqslant z_{i,r+1}, \quad i \in J, r = 2, \dots, G_i - 1. \tag{10}$$

Application: Sample Average Approximation for the PpCP

L.I. Martínez-Merino, M. Albareda-Sambola, A.M. Rodríguez-Chía.
The probabilistic p-center problem: Planning service for potential customers.
European Journal of Operational Research, 262:509-520, 2017



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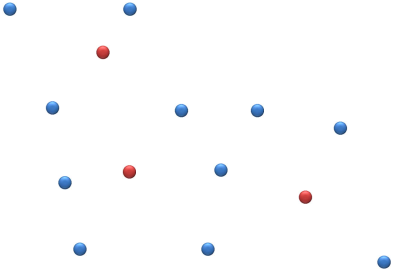

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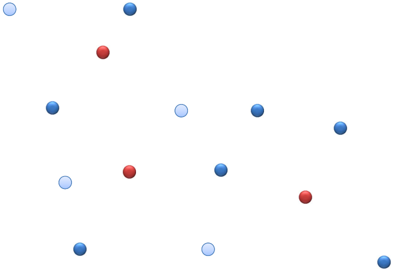

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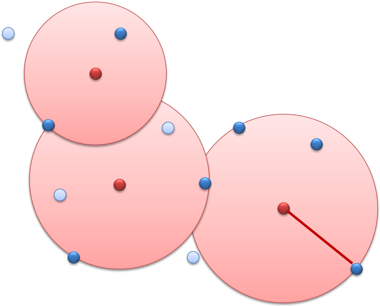

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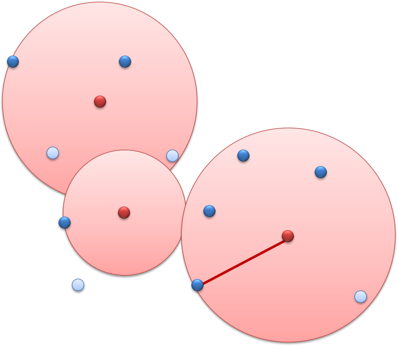

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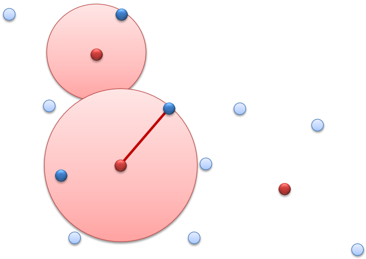

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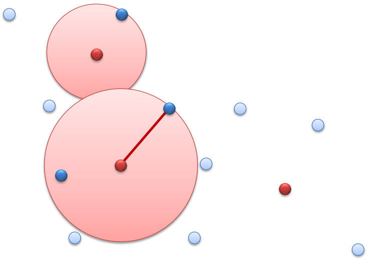
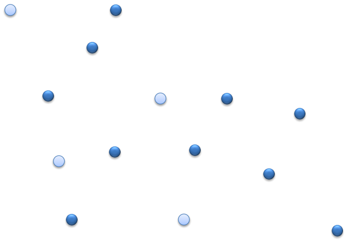


Objective: Minimize the expected maximum distance from a demand client to a facility.

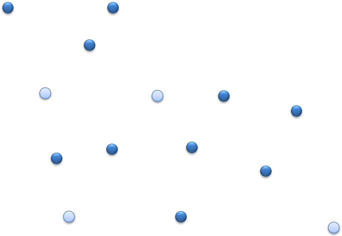
SpCP and PpCP

$$s_1 = (0,1,1,1,0,1,1,0,1,1,1,1,0,1)$$

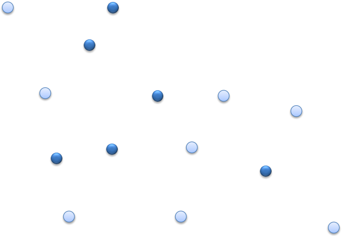


SpCP and PpCP

$$
\begin{aligned}
s_1 &= (0,1,1,1,0,1,1,0,1,1,1,1,0,1) \\
s_2 &= (1,1,1,0,0,1,1,1,1,1,1,0,1,0)
\end{aligned}
$$

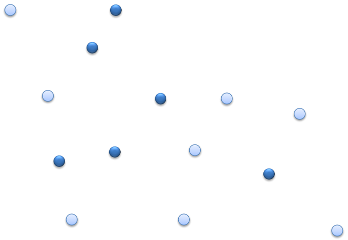


$S p$CP and $P p$CP



$$
\begin{aligned}
s_1 &= (0,1,1,1,0,1,1,0,1,1,1,1,0,1)\\
s_2 &= (1,1,1,0,0,1,1,1,1,1,1,0,1,0)\\
s_3 &= (0,1,1,0,1,0,0,1,1,0,1,0,0,0)\\
&\vdots
\end{aligned}
$$

S_pCP and P_pCP



$$
\begin{aligned}
s_1 &= (0,1,1,1,0,1,1,0,1,1,1,1,0,1)\\
s_2 &= (1,1,1,0,0,1,1,1,1,1,1,0,1,0)\\
s_3 &= (0,1,1,0,1,0,0,1,1,0,1,0,0,0)\\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
w_1 &= \pi_1\\
w_2 &= \pi_2\\
w_3 &= \pi_3\\
&\vdots
\end{aligned}
$$

Sample Average Approximation for the PpCP

$$(SP) \min_{x \in S} f_1(x) + \mathbb{E}_\xi v(x,\xi) \text{ where } v(x,\xi) = \min_{y \in U(x,\xi)} f_2(y,\xi)$$

Usually Ω is too large $\longrightarrow$ Use samples $\Omega^1, \Omega^2, \ldots$

$$z^k, x^k \leftarrow \min_{x \in S} f_1(x) + \sum_{s \in \Omega^k} \frac{1}{|\Omega^k|} f_2(x, y^s, \xi^s)$$

$$\text{s.t. } x \in S$$

$$y^s \in U(x, \xi^s) \qquad s \in \Omega^k$$

Proposition (Kleywegt et al., 2002)

When the sample size $N \to \infty$, the optimal optimal solution of SAA converges to the optimal solution of the original stochastic problem almost surely.

Computational results

- Submatrices and matrices of ORLIB p-median data.
  http://people.brunel.ac.uk/∼mastjjb/jeb/orlib/files/
  $n = 40, 50, 75, 100, 200, 300, 400,\ p = 3, 7, 10, 15, 20, 25, 30$
- Mosel/Xpress 8.0 used in all implementations.
- Intel(R) Core(TM) i7-4790K CPU 32 GB RAM.
- Time limit: 2 hours.

Comparison of formulations times

n	p	F1	F2	F3	F3-(1)+(2)	F3-(1)+(3)	F3-(1)+(4)	F3-(1)+(4)*
40	3	3.12	13.13	4.85	11.46	7.69	3.25	**2.75**
40	7	7.41	11.45	4.65	20.34	8.56	**2.62**	2.77
40	10	5.85	14.14	4.09	17.93	7.09	**2.28**	2.54
50	5	11.72	38.32	7.86	34.72	15.52	**5.46**	5.50
50	10	45.50	60.62	12.96	70.69	17.71	7.79	**5.94**
50	15	57.11	32.18	10.91	36.25	15.87	**6.54**	8.80
75	5	110.47	292.01	47.01	442.62	52.17	24.48	**18.73**
75	10	927.58	416.00	100.70	239.59	66.51	26.83	**19.97**
75	15	1973.03	380.97	59.88	201.43	73.25	28.21	**23.56**
100	10	5981.32(4)	1090.56	164.32	605.83	186.07	**63.53**	64.09
100	15	7200.00(5)	1611.93	142.57	435.76	271.07	77.88	**64.12**
100	25	5098.56(4)	1212.76	133.52	674.77	336.73	78.34	**64.62**

Comparison of formulations times

n	p	F1	F2	F3	F3-(1)+(2)	F3-(1)+(3)	F3-(1)+(4)	F3-(1)+(4)*
40	3	3.12	13.13	4.85	11.46	7.69	3.25	**2.75**
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$$
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z_{ir} &\geqslant 0, \quad i \in J, r = 3, \dots, G_i.
\end{aligned}
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n	p	F1	F2	F3-(1)+(4)*
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Comparison of formulations times

n	p	F1	F2	F3-(1)+(4)*	Binary*
40	3	3.12	13.13	2.75	1.13
40	7	7.41	11.45	2.77	1.06
40	10	5.85	14.14	2.54	1.37
50	5	11.72	38.32	5.50	2.29
50	10	45.50	60.62	5.94	2.99
50	15	57.11	32.18	8.80	3.63
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Comparison of formulations times

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100	15	7200.00(5)	1611.93	64.12	30.26
100	25	5098.56(4)	1212.76	64.62	28.95
200	10	-	-	1248.75	275.96
200	20	-	-	436.89	82.42
200	30	-	-	503.01	89.68

Reduction in the number of variables

n	p	%u fixed	F3-(1)+(4)* Time	F3-(1)+(4)* LP-Gap	Binary* t prepo	Binary* t total	Binary* LP Gap
75	5	**43.11**	18.73	54.79	0.51	**7.95**	**8.55**
75	10	**36.63**	19.97	59.35	0.49	**11.04**	**10.82**
75	15	**32.79**	23.56	64.43	0.46	**10.95**	**18.42**
100	10	**37.32**	64.09	59.06	1.14	**29.22**	**10.94**
100	15	**34.84**	64.12	61.87	0.95	**30.26**	**15.45**
100	25	**31.68**	64.62	69.13	0.91	**28.95**	**23.35**
200	10	**40.72**	1248.75	56.86	9.19	**275.96**	**8.95**
200	20	**35.33**	436.89	58.97	9.60	**82.42**	**11.04**
200	30	**33.96**	503.01	62.75	7.84	**89.68**	**15.28**

Valid inequalities

n	p	Binary*	(6)	(7)	(8)	(10)
75	5	**7.95**	12.61	8.02	8.02	11.95
75	10	11.04	13.95	**11.08**	11.11	16.25
75	15	10.95	12.96	11.09	**11.08**	14.24
100	10	29.22	37.42	29.21	**29.13**	43.85
100	15	**30.26**	45.75	30.29	30.28	46.58
100	25	28.95	37.38	28.82	**28.73**	44.35
200	10	275.96	539.47	275.58	**275.37**	289.21
200	20	**82.42**	161.46	82.49	82.69	92.31
200	30	**89.68**	176.37	90.28	90.04	120.44
300	15	**509.79**	1298.54	512.82	513.38	523.33
300	30	**315.13**	591.42	318.61	316.23	372.64
300	45	535.69	813.88	538.52	533.47	**442.23**
400	20	1017.28	3305.29	1011.01	1012.90	**722.30**
400	40	663.16	1863.28	666.45	**660.53**	805.02
400	60	475.14	1246.22	**474.36**	475.05	735.84

Results for ORLIB data instances

	n	p	t_{solv}	t_{prep}	t_{total}	LP Gap		n	p	t_{solv}	t_{prep}	t_{total}	LP Gap
pmed1	100	5	46.00	1.12	47.66	8.75	pmed21	500	5	120.12	334.01	466.05	4.86
pmed2	100	10	26.08	1.14	28.13	9.91	pmed22	500	10	253.44	307.79	597.42	7.78
pmed3	100	10	11.51	1.09	12.98	8.05	pmed23	500	50	7200.19	229.07	7447.45	18.36
pmed4	100	20	28.27	0.92	29.48	18.50	pmed24	500	100	1068.26	320.28	1392.83	22.16
pmed5	100	33	18.61	0.72	19.52	37.80	pmed25	500	167	1591.44	199.79	1793.49	40.16
pmed6	200	5	75.99	10.02	92.17	6.95	pmed26	600	5	259.63	677.29	959.90	4.58
pmed7	200	10	89.48	8.24	99.64	8.59	pmed27	600	10	427.28	593.22	1056.32	8.91
pmed8	200	20	279.66	8.63	290.36	13.40	pmed28	600	60	1784.88	572.43	2371.33	13.53
pmed9	200	40	48.49	8.75	57.95	17.09	pmed29	600	120	487.86	425.46	918.56	22.71
pmed10	200	67	47.67	6.26	54.21	33.24	pmed30	600	200	188.01	505.15	696.21	32.72
pmed11	300	5	30.98	44.26	78.77	5.05	pmed31	700	5	132.35	1318.46	1470.65	2.73
pmed12	300	10	57.64	41.64	105.20	6.77	pmed32	700	10	4250.54	788.62	5104.98	7.09
pmed13	300	30	786.58	33.21	829.07	17.75	pmed33	700	70	2666.15	1024.91	3729.87	13.66
pmed14	300	60	338.51	38.84	380.14	19.62	pmed34	700	140	305.98	752.26	1067.68	16.55
pmed15	300	100	60.15	40.86	101.79	27.72	pmed35	800	5	180.27	1429.91	1647.11	4.18
pmed16	400	5	47.96	114.38	165.70	3.88	pmed36	800	10	2376.87	1837.73	4283.38	6.03
pmed17	400	10	184.14	115.38	309.41	6.19	pmed37	800	80	5553.53	1353.61	6987.60	14.52
pmed18	400	40	384.20	86.96	480.31	9.80	pmed38	900	5	207.01	2844.10	3109.63	5.43
pmed19	400	80	3354.26	119.21	3476.32	21.85	pmed39	900	10	2993.05	2981.84	6021.92	7.82
pmed20	400	133	137.76	93.43	232.63	39.05	pmed40	900	90	7205.28	3780.23	11033.40	13.07

SAA for the PpCP

n	p	PpCP Time	F1 SAA Gap	F1 SAA Time	Binary* SAA Gap	Binary* SAA Time
30	3	**2.01**	0.00	13.90	0.00	11.27
30	7	13.61	0.14	12.78	0.15	**9.40**
30	10	22.99	0.00	**16.24**	0.00	16.54
40	3	**8.28**	0.00	40.90	0.00	19.94
40	7	148.22	0.01	98.39	0.20	**19.45**
40	10	295.52	0.01	96.68	0.01	**19.52**
50	5	243.17	0.03	162.76	0.00	**44.68**
50	10	4083.75	0.01	462.26	0.12	**67.74**
50	15	21782.53	0.21	794.07	0.01	**71.63**
75	5	4108.22	0.03	1386.77	0.03	**150.28**

n	p	PpCP Time	Gap_{BS}	SAA Time
75	10	> 86400	-2.55	200.57
75	15	> 86400	-5.67	258.32
100	10	> 86400	-10.25	491.73
100	15	> 86400	-15.31	449.02
100	25	> 86400	-20.40	850.55

Conclusions

- Introduction of a new extension of the p-center problem with demand points divided in different strata.
- Comparison of several formulations.
- Reduction in the number of variables and valid inequalities.
- Application of SpCP formulations in the SAA for obtaining good bounds on the PpCP.

Many thanks for your attention!