An exact algorithm for the Interval Transportation Problem

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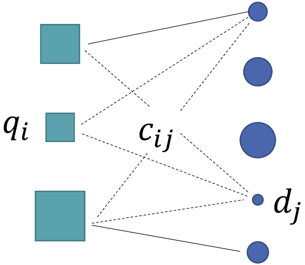
Outline

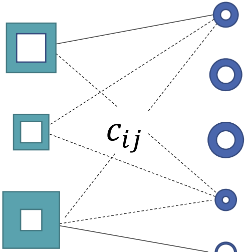
1. Interval Transportation Problem
2. Proposed algorithm
3. Can we do better?

Interval Transportation Problem



$t(q, d)$ = minimum total transportation cost satisfying q and d

Interval Transportation Problem



$$q_i \in [\underline{Q_i} \, ; \, \overline{Q_i}] \qquad d_j \in [\underline{D_j} \, ; \, \overline{D_j}]$$

$$t^* = \max t(q, d) \text{ (s.t. feasible)}$$

Interval Transportation Problem

- $\sum \underline{D}_j > \sum \bar{Q}_i$: all instances unfeasible.

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- We take

$$R = \{(q, d) \in [\underline{Q}; \bar{Q}] \times [\underline{D}; \bar{D}] : \sum_i q_i \geqslant \sum_j d_j\}.$$

The *interesting* instances are those with

$$\emptyset \neq R \subsetneq [\underline{Q}; \bar{Q}] \times [\underline{D}; \bar{D}]$$

Interval Transportation Problem

- x_{ij} : amount to ship from i to j

$$
\begin{aligned}
t^* = \mathrm{Max}\ & t(q,d) \\
\mathrm{s.t} & \sum_i q_i \geqslant \sum_j d_j, \\
& \underline{D} \leqslant d \leqslant \bar{D}, \\
& \underline{Q} \leqslant q \leqslant \bar{Q}.
\end{aligned}
$$

$$
\begin{aligned}
t(q,d) = \mathrm{Min}\ & \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} \\
\mathrm{s.t.}\ & \sum_{i\in I} x_{ij} \geqslant d_j \quad \forall j \in J, \\
& \sum_{j\in J} x_{ij} \leqslant q_i \quad i \in I, \\
& x_{ij} \geqslant 0, \qquad i \in I, j \in J.
\end{aligned}
$$

What has been done?

Liu, S.-T.
The total cost bounds of the transportation problem with varying demand and supply.
Omega 31:244-251, 2003.

Cerulli, R et al.
Best and worst values of the ooptimal cost of the interval transportaiton problem.
Springer Proceedings in Mathematics and Statistics 217:367-374, 2017.

What do we propose?

Inspired in:

Tsoukalas et al.,
A global optimization algorithm for generalaized semi-infinite, continuous minimax with cupled constraints and bi-level problems.
Journal of Global Optimization 44:235-250, 2009.

What do we propose? Main ingredients

- First, find UB, LB such that $t^* \in [LB, UB]$

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- Given $u_0 \in [LB, UB]$:
  $\mathtt{ATT}(u_0) \leftrightarrow$ does $(q, d) \in R$ exist with $t(q, d) \geqslant u_0$?
  (*i.e.*, $t^* \geqslant u_0$?)

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  (*i.e.*, $t^* \geqslant u_0$?)
- Binary search from $[LB, UB]$.

How to answer $\mathtt{ATT}(u_0)$

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How to answer $\mathtt{ATT}(u_0)$

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\begin{aligned}
P: \quad z^* = \ & \mathrm{Min}\ \theta \\
& \text{s.t. } \theta \geqslant u_0 - \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \\
& \textstyle\sum_i q_i \geqslant \sum_j d_j, \\
& \underline{D} \leqslant d \leqslant \bar{D}, \\
& \underline{Q} \leqslant q \leqslant \bar{Q}. \\
& \textstyle\sum_i x_{ij} \geqslant d_j & \forall j \in J, \\
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$$z^* \leqslant 0 \iff \mathtt{ATT}(u_0)$$

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RP: \quad z^* = \ & \mathrm{Min}\ \theta \\
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& \textstyle\sum_i x_{ij} \geqslant d_j && \forall j \in J, \\
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& x \text{ cheaper than} && \textbf{any } x \text{ feasible for } q, d \textbf{ in Y}
\end{aligned}
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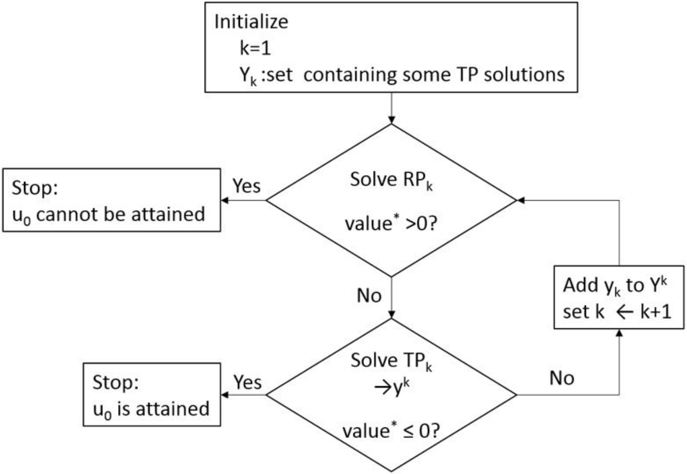
$$z^* \leqslant 0 \Longleftarrow \mathtt{ATT}(u_0)$$

Iterative procedure to answer $\mathtt{ATT}(u_0)$



Restricted problem RP_k

$$z^* = \min_{\substack{(d,q)\in R \\ x \text{ feasible for } (d,q)}} \max_{y^\ell \in Y^k} \max \left\{ u_0 - \sum_{i\in I} \sum_{j\in J} c_{ij} x_{ij}, \Delta^\ell \right\},$$

where

$$\Delta^\ell = \min \left\{ \sum_{i,j} c_{ij}(x_{ij} - y_{ij}^\ell), \min_{i\in I} \left\{ q_i - \sum_{j\in J} y_{ij}^\ell + \epsilon \right\}, \min_{j\in J} \left\{ \sum_{i\in I} y_{ij}^\ell - d_j + \epsilon \right\} \right\}.$$

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- 1+n+m binaries to identify Δ^ℓ for each solution ℓ

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- Formulated as a MILP
- 1+n+m binaries to identify Δ^ℓ for each solution ℓ
- big-M constraints to set $\theta \geqslant \Delta^\ell$
- $z^* \leqslant 0 \Rightarrow$ either $\mathtt{ATT}(u_0)$ or Y^k needs update (TP for d^k, q^k)

Main drawback

Slight modifications of (d, q) yield most y^{ℓ} solutions unfeasible

$$\Downarrow$$

Many (many!) iterations needed to answer $\mathtt{ATT}(u_0)$

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According to our experience, especially true if $UB >> t^*$ and $t^* < u_0$

Our algorithm. v2

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- **Idea:** Consider visited basis in Y^k instead of particular solutions.

Updated RP$_k$

$$z^* = \min_{\substack{(d,q)\in R \\ x \text{ feasible for } (d,q)}} \max_{y^\ell \in Y^k} \max \left\{ u_0 - \sum_{i\in I} \sum_{j\in J} c_{ij} x_{ij}, \Delta^\ell \right\},$$

where

$$\Delta^\ell = \min \left\{ \sum_{i,j} c_{ij} x_{ij} + \sum_{i\in I} \alpha_i^\ell q_i - \sum_{j\in J} \beta_j^\ell d_j, \right.$$
$$\left. \min_{1\leqslant r\leqslant n+m} \left\{ -\sum_{i\in I} a_{ri}^\ell q_i + \sum_{j\in J} a_{r(n+j)}^\ell d_j + \epsilon \right\} \right\}.$$

Updated RP_k

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Updated RP_k

- Again, MILP formulation using binaries to identify Δ^ℓ.
- New big-M constraints to force $\theta \geqslant \Delta^\ell$
- Fewer iterations needed to answer $\texttt{ATT}(u_0)$

Final comments

- Ongoing computational experience. First observations:

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  - Surprisingly, very different instances obtained in consecutive iterations of TP^k (1)
  - TP^k solutions typically far from optimal for TP
  - v2: much fewer iterations, but get expensive fast

Final comments

- Ongoing computational experience. First observations:
  - Surprisingly, very different instances obtained in consecutive iterations of TP^k (1)
  - TP^k solutions typically far from optimal for TP
  - v2: much fewer iterations, but get expensive fast
- LB initialization:
  - v1: Several TP's (cheapest, max. throughput, +random)
  - v2: One single TP (max. throughput)
- UB initialization:
  - max. dem instance + dummy agent

Thanks!