Bilevel programming models for multi-product location problems

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Motivation

- An important decision that retailers must take concerns their product assortment.
- If the firm does not consider the clients' preferences, the rotation rate of certain products could decrease.
- Therefore, the retailers must reduce significantly the prices of some products to reduce their inventory.
- Most of the literature considers assortment in a single store (Kök, A Gürhan and Fisher (2008)).
- The literature does not consider the client transportation costs to reach the stores.

Description problem

The retailer's problem

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- A set $\mathcal{K}$ of products to a price π_k.
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The customer's problem

- Reservation price r_{ik} of client $i \in \mathcal{I}$ for the product $k \in \mathcal{K}$.
- Transportation cost d_{ij} of client $i \in \mathcal{I}$ to store $j \in \mathcal{J}$.
- Utility (benefit) for $i \in \mathcal{I}$ from purchasing product $k \in \mathcal{K}$, with discount level l at store $j \in \mathcal{J}$ is:

$$b_{ijkl} = r_{ik} - \pi_k \alpha_{jkl} - 2 \cdot d_{ij}$$

- Each client buys at most one product.

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- Each client buys at most one product.
- **Each customer buys a product from a store that both maximize his utility or buys nothing**

Review literature

No paper considering capacity + pricing + multi-store + distance

- Green and Krieger (1985, 1989) product allocation. Heuristics.
- Ghoniem and Madah (2015) assortment + pricing. Customer segments. No space limitations.
- Ghoniem et al. (2016) assortment + pricing. Multiple product categories.
- Besbes & Sauré (2015); Moon et al (2017); Hubner & Schaal (2017), Aras & Kukaydin (2017)

Model

Bilevel linear model

Leader's problem

$$\max \sum_{i \in \mathcal{I}} \sum_{(j,k,l) \in \mathcal{T}_i} \alpha_{jkl} \pi_j x_{ijkl} \qquad (1)$$

subject to

$$\sum_{k \in \mathcal{K}} \sum_{l \in \mathcal{L}} y_{jkl} \leq p_j \qquad (j \in \mathcal{J}) \qquad (2)$$

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subject to

$$\sum_{k \in \mathcal{K}} \sum_{l \in \mathcal{L}} y_{jkl} \leq p_j \qquad (j \in \mathcal{J}) \tag{2}$$

Follower's problem

For all $i \in \mathcal{I}$

$$\max \sum_{(j,k,l) \in \mathcal{T}_i} b_{ijkl} x_{ijkl} \tag{3}$$

subject to

$$x_{ijkl} \leq y_{jkl} \quad ((j,k,l) \in \mathcal{T}_i), \tag{4}$$

$$\sum_{(j,k,l) \in \mathcal{T}_i} x_{ijkl} \leq 1 \tag{5}$$

Model

One level

- Replace follower problem by the primal constraints and optimality constraints.
- Define for all i :

$$\mathcal{B}_{ijkl} = \{(j',k',l') \in \mathcal{T}_i | b_{ijkl} \leq b_{i'j'k'l'}\} \quad \text{and} \quad \mathcal{W}_{ijkl} = \mathcal{T}_i \setminus \mathcal{B}_{ijkl}$$

- The optimality constraint can be expressed as :

$$\sum_{(j',k',l') \in \mathcal{B}_{ijkl}} x_{ij'k'l'} \geq y_{jkl} \quad (i \in \mathcal{I}, (j,k,l) \in \mathcal{T}_i) \tag{6}$$

or

$$y_{jkl} + \sum_{(j',k',l') \in \mathcal{W}_{ijkl}} x_{ij'k'l'} \leq 1 \quad (i \in \mathcal{I}, (j,k,l) \in \mathcal{T}_i) \tag{7}$$

or

$$\sum_{(j',k',l') \in \mathcal{W}_{ijkl}} x_{ij'k'l'} + \sum_{(j',k',l') \in \mathcal{W}_{i'jkl} \cap \mathcal{T}_i \setminus \mathcal{W}_{ijkl}} x_{i'j'k'l'} + y_{jkl} \leq 1 \tag{8}$$

$$(i \in \mathcal{I}, i' \in \mathcal{I}, (j,k,l) \in \mathcal{T}_i)$$

Model

Single level formulation. Without optimality constraint

$$\max \sum_{i \in \mathcal{I}} \sum_{(j,k,l) \in \mathcal{T}_i} \alpha_{jkl} \pi_j x_{ijkl} \tag{9}$$

$$\text{subject to} \quad \sum_{k \in \mathcal{K}} \sum_{l \in \mathcal{L}} y_{jkl} \leq p_j \qquad (j \in \mathcal{J}), \tag{10}$$

$$x_{ijkl} \leq y_{jkl} \qquad (i \in \mathcal{I}, (j,k,l) \in \mathcal{T}_i), \tag{11}$$

$$\sum_{(j,k,l) \in \mathcal{T}_i} x_{ijkl} \leq 1 \qquad (i \in \mathcal{I}), \tag{12}$$

$$x_{ijkl} \in \{0,1\} \qquad (i \in \mathcal{I},\ (j,k,l) \in \mathcal{T}_i), \tag{13}$$

$$y_{jkl} \in \{0,1\} \qquad (j \in \mathcal{J},\ k \in \mathcal{K},\ l \in \mathcal{L}). \tag{14}$$

Comparison of the customer preference constraints

Now let M0 be the model without the customer preference constraints. For each customer preference constraints we can define a different model:

- M1 = model M0 with constraints (7).
- M2 = model M0 with constraints (8).
- M3 = model M0 with constraints (6).

Results comparison of models

Table: Instances with I=30, J=5, K=50, L=3, Cap=5

Name	#Const			GAP			LP Time			Mip Time		
	M1	M2	M3	M1	M2	M3	M1	M2	M3	M1	M2	M3
30-50-1	13324	56016	13385	1,7	0,5	1,7	0,7	211,8	2,5	15,4	1012,3	65,3
30-50-2	11472	70067	11537	1,0	0,0	1,0	0,6	42,4	2,4	6,5	1441,5	34,1
30-50-3	14817	42950	14880	1,2	0,1	1,2	1,0	32,9	3,4	10,1	1338,9	70,6
30-50-4	11620	72120	11683	2,0	0,2	2,0	0,6	45,1	6,3	14,6	2410,7	102,5
30-50-5	14905	78998	14969	1,6	0,0	1,6	0,8	41,3	2,6	8,9	1667,9	60,1
30-50-6	15448	69176	15513	0,6	0,6	0,6	0,6	250,0	2,3	5,9	2180,1	20,9
30-50-7	14648	48927	14711	2,9	0,2	2,9	1,0	6,9	4,8	27,5	547,2	164,5
30-50-8	12119	47008	12183	1,2	0,4	1,2	0,4	6,7	2,4	5,6	265,3	24,4
30-50-9	12187	69958	12249	1,3	0,0	1,3	0,8	8,9	6,1	9,1	344,2	33,5
30-50-10	14506	41670	14573	0,8	0,2	0,8	0,5	22,8	2,3	5,9	467,4	23,2
Average	13505	59689	13568	1,4	0,2	1,4	0,7	66,9	3,5	11,0	1167,5	59,9

From the experiments we have to:

- M1 gets a lower $Time\ MIP$ and $Time\ LP$ than M3.
- M2 is the most adjusted.
- M2 is the one that generates the most restrictions.

Model

First Lagrangian relaxation

$$\max \sum_{i \in \mathcal{I}} \sum_{(j,k,l) \in \mathcal{T}_i} \alpha_{jkl} \pi_j x_{ijkl} \tag{7}$$

$$\text{subject to} \quad \sum_{k \in \mathcal{K}} \sum_{l \in \mathcal{L}} y_{jkl} \leq p_j \qquad (j \in \mathcal{J}), \tag{8}$$

$$\boxed{\sum_{(j',k',l') \in \mathcal{W}_{ijkl}} x_{ij'k'l'} + y_{jkl} \leq 1} \qquad (i \in \mathcal{I}, (j,k,l) \in \mathcal{T}_i), \tag{9}$$

$$x_{ijkl} \leq y_{jkl} \qquad (i \in \mathcal{I}, (j,k,l) \in \mathcal{T}_i), \tag{10}$$

$$\sum_{(j,k,l) \in \mathcal{T}_i} x_{ijkl} \leq 1 \qquad (i \in \mathcal{I}), \tag{11}$$

Model

Second Lagrangian relaxation

$$\max \sum_{i \in \mathcal{I}} \sum_{(j,k,l) \in \mathcal{T}_i} \alpha_{jkl} \pi_j x_{ijkl} \tag{7}$$

$$\text{subject to} \quad \sum_{k \in \mathcal{K}} \sum_{l \in \mathcal{L}} y_{jkl} \leq p_j \qquad (j \in \mathcal{J}), \tag{8}$$

$$\sum_{(j',k',l') \in \mathcal{W}_{ijkl}} x_{ij'k'l'} + y_{jkl} \leq 1 \qquad (i \in \mathcal{I}, (j,k,l) \in \mathcal{T}_i), \tag{9}$$

$$x_{ijkl} \leq y_{jkl} \qquad (i \in \mathcal{I}, (j,k,l) \in \mathcal{T}_i), \tag{10}$$

$$\sum_{(j,k,l) \in \mathcal{T}_i} x_{ijkl} \leq 1 \qquad (i \in \mathcal{I}), \tag{11}$$

Model

Third Lagrangian relaxation

$$\max \sum_{i\in\mathcal{I}} \sum_{(j,k,l)\in\mathcal{T}_i} \alpha_{jkl}\pi_j x_{ijkl} \tag{7}$$

$$\text{subject to} \quad \sum_{k\in\mathcal{K}} \sum_{l\in\mathcal{L}} y_{jkl} \le p_j \qquad (j\in\mathcal{J}), \tag{8}$$

$$\boxed{\sum_{(j',k',l')\in\mathcal{W}_{ijkl}} x_{ij'k'l'} + y_{jkl} \le 1} \qquad (i\in\mathcal{I}, (j,k,l)\in\mathcal{T}_i), \tag{9}$$

$$x_{ijkl} \le y_{jkl} \qquad (i\in\mathcal{I}, (j,k,l)\in\mathcal{T}_i), \tag{10}$$

$$\boxed{\sum_{(j,k,l)\in\mathcal{T}_i} x_{ijkl} \le 1} \qquad (i\in\mathcal{I}), \tag{11}$$

Obtaining Feasible Solutions

- In the three cases, the values of $\{y^*\}$ are feasible for the constraints of the retailer.
- Let V be the set of tuples (j, k, l) selected by the retailer, i.e. $\mathcal{Y} = \{(j, k, l) | y^*_{jkl} = 1\}$.
- So, each customer selects the $(j, k, l) \in \mathcal{V}$ that maximize his utility.
- We slightly improve the solution found through local search (over y^*_{jkl}).

Preliminary Computational Results

Test problem

INDEX	SET 1	SET 2	SET 3	SET 4
CLIENTS	200	200	300	300
MALLS	4	4	4	4
PRODUCTS	80	150	80	150
LEVELS	3	3	3	3
CAPACITIES	5	5	5	5

Table: Description of the test problems

- Stopping criteria
  - 3600 seconds.

Computational Results

Rate	Set	GUROBI	LR1	LR2	LR3
$\frac{100 \cdot (z_{LP} - z_{UB})}{z_{LP}}$	SET 1	1.31	0.40	0.00	0.00
	SET 2	0.31	0.98	0.00	0.00
	SET 3	1.09	0.89	0.00	0.00
	SET 4	0.24	1.95	0.00	0.00
$\frac{100 \cdot (z_{LP} - z_{LB})}{z_{LP}}$	SET 1	4.45	5.92	11.83	12.20
	SET 2	8.38	5.76	15.00	14.80
	SET 3	5.16	6.57	12.98	13.40
	SET 4	19.17	6.99	16.20	16.30
$\frac{100 \cdot (z_{UP} - z_{LB})}{z_{LB}}$	SET 1	3.17	5.49	11.83	12.20
	SET 2	8.05	4.72	15.00	14.80
	SET 3	4.01	5.62	12.98	13.40
	SET 4	19.48	4.90	16.20	16.30

Table: Main results : where z_{LB} be the best lower bound found; z_{UB} be the best upper bound found and Z_{LP} be the linear relaxation of the complete problem

Summary results

- The first Lagrangian relaxation obtained the best gap between the three Lagrangian relaxations.
- In the medium instances GUROBI get better results.
- In the large instances Lagrangian relaxation get better results.

Conclusion and the future work

- For large instances, the Lagrangian relaxation gives a better GAP that Branch & Bound.
- How to improve the bounds ? Branch and cut for M2?
- We are currently working on a new (two index) formulation of the problem.
- A related problems: each customer can choose a bundle of products.
- In this case, the clients could buy all in the same store or in different stores.

Thanks for your attention!