IX IWOLOCA: On location-allocation problems for dimensional facilities

Lina Mallozzi[1] Justo Puerto[2] **Moisés Rodríguez-Madrena**[2]

[1] University of Naples Federico II, Naples, Italy
[2] IMUS, Universidad de Sevilla, Seville, Spain

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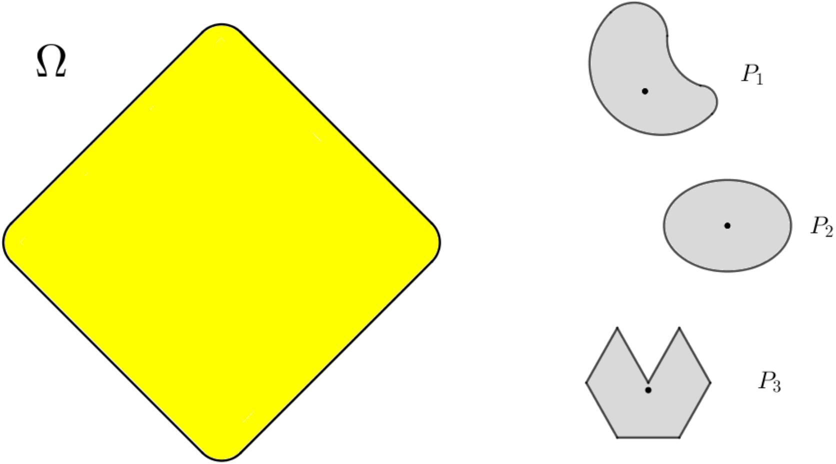
Introduction

Locating dimensional facilities in a region



Locating dimensional facilities in a region

We are given Ω, a Borel, compact subset of $\mathbb{R}^2$, that represents a demand region. We assume that customers in Ω are distributed according to a demand density $D \in \mathcal{L}^2(\Omega)$ that is an absolutely continuous probability measure, where $D : \Omega \to \mathbb{R}$ is a nonnegative function with unit integral

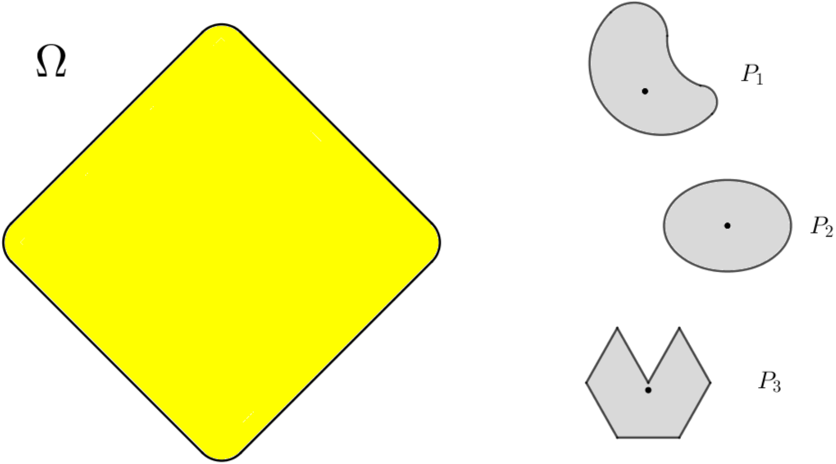
$$\int_{\Omega} D(q)dq = 1,$$

being $q = (x, y) \in \Omega$ and $dq = dxdy$.

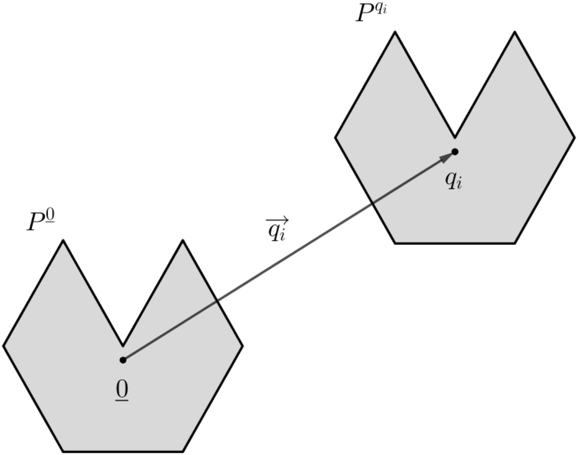
The goal is to locate ρ given compact sets $P_1, ..., P_\rho$ ($\rho \in \mathbb{N}$) in Ω, representing some service centers with dimensional extension.

Locating dimensional facilities in a region

What do we consider a suitable solution (location)?



Locating dimensional facilities in a region



Locating dimensional facilities in a region

Definition 1 (suitable solution)

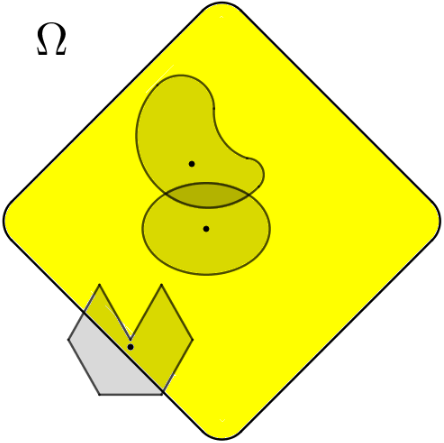
The point $Q = (q_1, ..., q_\rho) \in \mathbb{R}^{2\rho}$ is a suitable solution if:

1. $P_i^{q_i} \subseteq \Omega$, for each $i \in \{1, ..., \rho\}$,
2. $\text{int}(P_i^{q_i}) \cap \text{int}(P_j^{q_j}) = \emptyset$, for all $i, j \in \{1, ..., \rho\}$ with $i \neq j$.

We denote by Γ the set of all the suitable solutions.

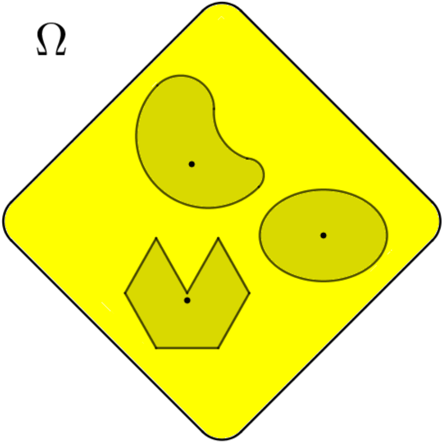
Locating dimensional facilities in a region

Not a suitable solution.



Locating dimensional facilities in a region

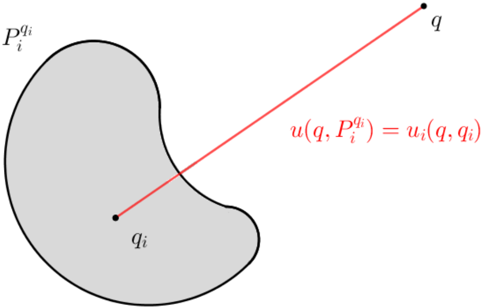
A suitable solution.



A bilevel model and existence of optimal solutions

Bilevel approach

We consider that the utility u paid from a point $q \in \mathbb{R}^2$ with respect to P_i is given by a continuous function $u_i : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ that depends on q and the location $q_i \in \mathbb{R}^2$ of P_i:



Bilevel approach

Particular cases of choosing the utility u in this way:

- Service point case:

$$u(q, P_i^{q_i}) = f_i(\gamma_i(q - q_i)).$$

- Utility induced by the Minkowski functional:

$$u(q, P_i^{q_i}) = \begin{cases} f_i(\gamma_{P_i}(q - q_i) - 1), & \text{if } q \notin P_i^{q_i}, \\ f_i(0), & \text{if } q \in P_i^{q_i}. \end{cases}$$

- Conservative planner (Brazil, Ras & Thomas (2014)):

$$u(q, P_i^{q_i}) = f_i\left(\max_{\tilde{q} \in P_i^{q_i}} \gamma_i(q - \tilde{q})\right).$$

Where $f_i : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\gamma_i : \mathbb{R}^2 \to \mathbb{R}$ is a norm, and $\gamma_{P_i} : \mathbb{R}^2 \to \mathbb{R}$ is the gauge induced by P_i (provided that it is well-defined).

Bilevel approach

In the spirit of a *social planner*, we are interested in finding a partition $A(Q) = (A_1(Q), ..., A_\rho(Q))$ of the customers in $\Omega(Q)$ solving the problem:

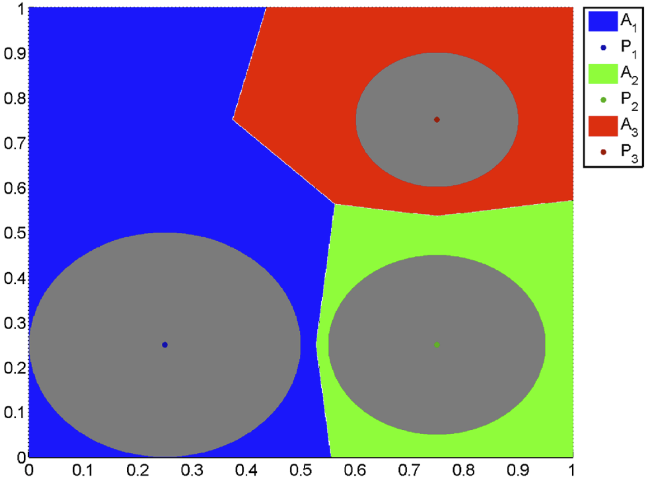
$$\min_{A(Q)\in\mathcal{A}_\rho(Q)} \sum_{i=1}^{\rho} \left[\int_{A_i(Q)} \left[a_i + u(q, P_i^{q_i})\right] D(q)dq \right], \qquad \mathsf{LL}(Q)$$

where $a_i > 0$ is the cost incurred by each customer to access facility P_i per unit demand. Note that:

$$\int_{A_i(Q)} \left[a_i + u(q, P_i^{q_i})\right] D(q)dq = \underbrace{a_i \int_{A_i(Q)} D(q)dq}_{\text{total access cost for facility } P_i} + \underbrace{\int_{A_i(Q)} u(q, P_i^{q_i}) D(q)dq.}_{\text{distribution cost in the service region } A_i(Q)}$$

Bilevel approach

Example 4.1 in Mallozzi & Puerto (2018).



Bilevel approach

Problem BL:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{\rho}\Bigg[\overbrace{I_i\left(\int_{P_i^{q_i}} B(q)dq\right)}^{\text{installation cost}} + \overbrace{C_i\left(\int_{\widehat{A}_i(Q)} D(q)dq\right)}^{\text{congestion cost}}\Bigg] + \overbrace{L\left(\int_{\bigcup_{i=1}^{\rho} P_i^{q_i}} D(q)dq\right)}^{\text{lost demand cost}} \\
s.t. \quad & \widehat{A}(Q) \in \arg \min_{A(Q)\in\mathcal{A}_{\rho}(Q)} \sum_{i=1}^{\rho}\left[\int_{A_i(Q)} \left[a_i + u(q, P_i^{q_i})\right] D(q)dq\right], \\
& Q \in \Gamma,
\end{aligned}
$$

where

- $B \in \mathcal{L}^2(\Omega)$ models the base installation cost,
- $I_1, ..., I_{\rho}, C_1, ..., C_{\rho}, L$ are non-decreasing, bounded, continuous, real valued functions.

Resolution via optimal transport mass

Mallozzi & Passarelli di Napoli (2015). Mallozzi & Puerto (2018). Carlier & Mallozzi (2018).

Theorem 1

Let $Q = (q_1, ..., q_\rho) \in \Gamma$. Suppose that the set

$$\{q \in \Omega(Q) : a_i + u(q, P_i^{q_i}) = a_j + u(q, P_j^{q_j})\} \tag{1}$$

is D-negligible, for all $i, j \in \{1, ..., \rho\}$ with $i \neq j$.

Then the problem LL(Q) admits a unique solution $A(Q) = (A_1(Q), ..., A_\rho(Q))$ that verifies

$$A_i(Q) = \{q \in \Omega(Q) : a_i + u(q, P_i^{q_i}) < a_j + u(q, P_j^{q_j}), \forall j \neq i\} \tag{2}$$

for each $i \in \{1, ..., \rho\}$, where the equalities are intended up to D-negligible sets.

Resolution via optimal transport mass

Proof of existence of optimal partition (outline):

$$\inf_{A(Q)\in\mathcal{A}_\rho(Q)} \sum_{i=1}^{\rho} \left[\int_{A_i(Q)} \left[a_i + u(q, P_i^{q_i})\right] D(q)dq \right]$$

$$= \inf_{\substack{T_\sharp \tilde{\mu} = \nu(\omega) \\ \omega \in S}} \int_{\Omega(Q)} c(q, T(q)) d\tilde{\mu}(q)$$

$$= \inf_{\omega \in S} \left\{ \mathcal{W}_c(\tilde{\mu}, \nu(\omega)) + \sum_{i=1}^{\rho} a_i \omega_i \right\},$$

where $S = \{\omega = (\omega_1, ..., \omega_\rho) \in \mathbb{R}^\rho : \omega_i \geq 0, \sum_{i=1}^{\rho} \omega_i = 1\}$ and $\nu(\omega) = \sum_{i=1}^{\rho} \omega_i \delta_{\tilde{q}_i}$.

Resolution via optimal transport mass

Assuming (1) for each $Q \in \Gamma$, we can define the *best reply function*

$$\widehat{A} : Q \in \Gamma \to \widehat{A}(Q) \in \mathcal{A}_\rho(Q),$$

that maps to a given suitable solution, the optimal partition of the customers given in (2).

Resolution via optimal transport mass

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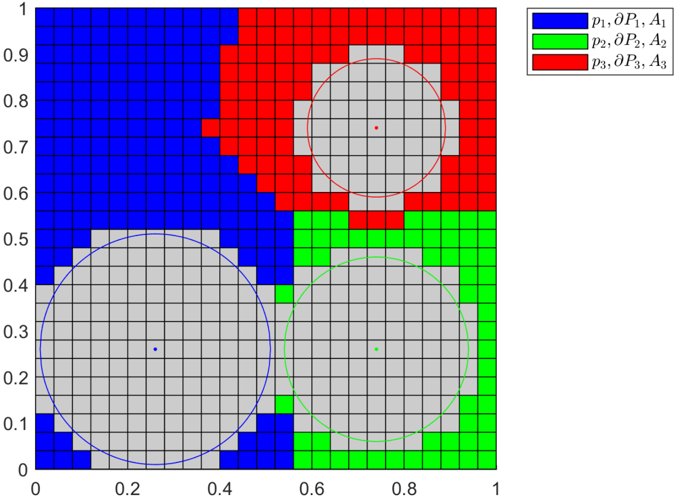
that maps to a given suitable solution, the optimal partition of the customers given in (2).

Theorem 2

There exists an optimal solution for the problem BL.

A convergent discrete approximation scheme

A convergent discrete approximation scheme



A convergent discrete approximation scheme

Theorem 3

Suppose that, for any suitable solution $Q \in \Gamma$ of the problem BL and for any $\delta > 0$, there exists $\tilde{Q} = (\tilde{q}_1, ..., \tilde{q}_\rho) \in \mathcal{B}_\infty(Q, \delta) \cap \Gamma$ such that $P_i^{\tilde{q}_i} \cap \partial\Omega = \emptyset$ for all $i \in \{1, ..., \rho\}$ and $P_i^{\tilde{q}_i} \cap P_j^{\tilde{q}_j} = \emptyset$ for all $i, j \in \{1, ..., \rho\}$ with $i \neq j$.

Then, for any $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$ such that:

1. $|\mathcal{F}(Q^*) - \boldsymbol{\mathcal{F}}(\boldsymbol{Q}^*, n)| < \varepsilon$,
2. $\left|\mathcal{F}(Q^*) - \mathcal{F}(\breve{Q})\right| < \varepsilon$,

for all $n \in \mathbb{N}$ with $n \geq n(\varepsilon)$, being

- Q^* an optimal suitable solution of the problem BL,
- $\boldsymbol{Q}^*$ an optimal suitable solutions of the problem DBL(n),
- $\breve{Q}$ the suitable solution of the problem BL codified by $\boldsymbol{Q}^*$.

Solution approaches

A mathematical programming formulation

We use the following sets and parameters to build our MILP model. For each $i \in \{1, ..., \rho\}$ and each $(r, s), (k, l) \in \mathbf{\Omega}$ (set of cells of the discretization), we define:

- $\mathbf{\Omega}_i = \{(k, l) \in \mathbf{\Omega} : P_i^{q(k,l)} \subseteq \Omega\}$,
- $\boldsymbol{E}_{rs}^i = \{(k, l) \in \mathbf{\Omega}_i : (r, s) \in \boldsymbol{P}_i^{(k,l)}\}$,
- $w_{rs}^D = \int_{(r,s)} D(q)dq$,
- $w_{rs}^B = \int_{(r,s)} B(q)dq$,
- $u_{rs,kl}^i = \begin{cases} u(q_{(r,s)}, P_i^{q(k,l)}), & \text{if } (r, s) \notin \boldsymbol{P}_i^{(k,l)}, \\ -a_i, & \text{if } (r, s) \in \boldsymbol{P}_i^{(k,l)}. \end{cases}$

A mathematical programming formulation

For each $i \in \{1, ..., \rho\}$ and each $(k, l), (r, s) \in \mathbf{\Omega}$, we use the following decision variables:

- $\theta^i_{kl} = \begin{cases} 1, & \text{if } p_i \text{ is located at the center of } (k, l), \\ 0, & \text{otherwise,} \end{cases}$
- $\tau^i_{rs} = \begin{cases} 1, & \text{if demand in } (r, s) \text{ is served by } P_i, \\ 0, & \text{otherwise,} \end{cases}$
- $\varphi_{rs} = \begin{cases} u(q_{(r,s)}, P_i), & \text{if } P_i \text{ serves the demand in } (r, s), \\ 0, & \text{if } (r, s) \text{ belongs to a cell facility.} \end{cases}$

A mathematical programming formulation

$$\min \quad \sum_{i=1}^{\rho} \overline{I_i^{\mathbf{PL}}} \left(\sum_{(r,s)\in\boldsymbol{\Omega}} \sum_{(k,l)\in\boldsymbol{E}_{rs}^i} w_{rs}^B \theta_{kl}^i \right) + \sum_{i=1}^{\rho} \overline{C_i^{\mathbf{PL}}} \left(\sum_{(r,s)\in\boldsymbol{\Omega}} w_{rs}^D \tau_{rs}^i \right) + \overline{L^{\mathbf{PL}}} \left(\sum_{(r,s)\in\boldsymbol{\Omega}} w_{rs}^D \left[1 - \sum_{i=1}^{\rho} \tau_{rs}^i \right] \right) \tag{3}$$

s.t. $SCDVPL$ (see Fourer (1985)), (4)

$$\sum_{(k,l)\in\boldsymbol{\Omega}_i} \theta_{kl}^i = 1, \quad \forall i \in \{1, \dots, \rho\}, \tag{5}$$

$$\sum_{i=1}^{\rho} \tau_{rs}^i + \sum_{i=1}^{\rho} \sum_{(k,l)\in\boldsymbol{E}_{rs}^i} \theta_{kl}^i = 1, \quad \forall (r,s) \in \boldsymbol{\Omega}, \tag{6}$$

$$\sum_{j=1}^{\rho} a_j w_{rs}^D \tau_{rs}^j + w_{rs}^D \varphi_{rs} \leq a_i w_{rs}^D + \sum_{(k,l)\in\boldsymbol{\Omega}_i} w_{rs}^D u_{rs,kl}^i \theta_{kl}^i, \quad \forall (r,s) \in \boldsymbol{\Omega},\ i \in \{1, \dots, \rho\}, \tag{7}$$

$$\varphi_{rs} \gtreqless \sum_{(k,l)\in\boldsymbol{\Omega}_i} u_{rs,kl}^i \theta_{kl}^i \mp M(1 - \tau_{rs}^i), \quad \forall (r,s) \in \boldsymbol{\Omega},\ i \in \{1, \dots, \rho\}. \tag{8}$$

Heuristic method

The heuristic proposed is a GRASP (see Feo & Resende (1995)) in which we can distinguish three modules:

1. WAVE_DIMFAC: is a continuous wavefront algorithm, it is able to generate a random suitable solution $Q \in \Gamma$ of the problem DBL from a given initial point.
2. GREEDY_DIMFAC: is a greedy algorithm that, given a suitable solution $Q \in \Gamma$ of the problem DBL, locally searches for another feasible solution improving the objective value of the first one.
3. GRASP_DIMFAC: is actually the GRASP, it works with a list of suitable solutions of the problem DBL obtained with the modules above.

Test experiments

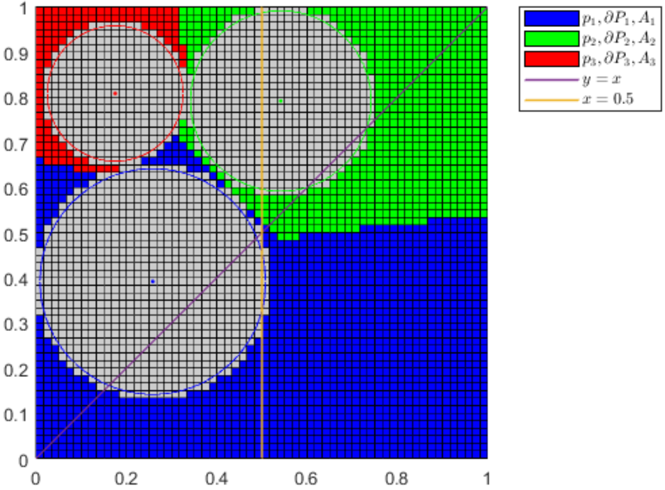
Test experiments

Test experiment 1:

- Example 4.1 in Mallozzi & Puerto (2018).
- $D(q) = \begin{cases} 8(x-0.5), & \text{if } x \geq 0.5, \\ 0, & \text{if } x < 0.5. \end{cases}$
- Regular grid with 60×60 cells.
- $B(q) = \begin{cases} 6(x-y), & \text{if } x \geq y, \\ 0, & \text{if } x < y. \end{cases}$
- $I_1 = I_2 = I_3 = \text{id},\ C_1 = C_2 = C_3 = 0,\ L = \text{id}.$

Test experiments

Solution obtained with the GRASP algorithm.



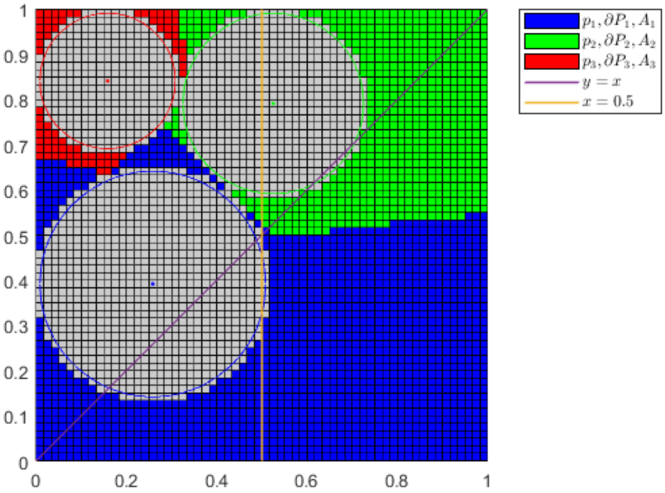
Test experiments

Performance of the GRASP algorithm:

- CPU time (preprocess+execution): 11336s+515s $\simeq$ 3.29h.
- Best objective value found: 0.1109.

Test experiments

Solution obtained with the MILP formulation.



Test experiments

Performance of the GRASP algorithm:

- CPU time (preprocess+execution): 11336s+515s $\simeq$ 3.29h.
- Best objective value found: 0.1109.

Performance of the MILP formulation (4h):

- CPU time (preprocess+execution): 469s + 3306s $\simeq$ 1.05h.
- Best objective value found: 0.0995.
- GAP (%): 0.

Test experiments

Test experiment 2:

- P_1 is a non-convex polygon and $u_1(q, q_1) = 0.8 \max_{\tilde{q} \in P_1^{q_1}} \ell_2(q, \tilde{q})$.
- P_2 is a regular pentagon and $u_2(q, q_2) = \ell_2(q, q_2)$.
- P_3 is the unit ball of a weighted Euclidean norm and $u_3(q, q_3) = 0.2\gamma_{P_3}(q - q_3)$.
- Regular grid with 60×60 cells.
- $a_1 = a_2 = a_3 = 1$.

Test experiments

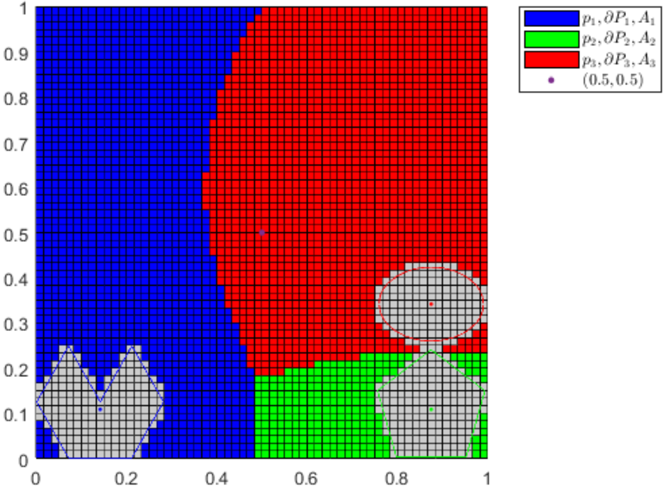
Test experiment 2:

- $D(q) = 1$.
- $B(q) = \begin{cases} 2(x+y), & \text{if } x \leq 0.5 \text{ and } y \leq 0.5, \\ 2(x+1-y), & \text{if } x \leq 0.5 \text{ and } y > 0.5, \\ 2(1-x+y), & \text{if } x > 0.5 \text{ and } y \leq 0.5, \\ 2(1-x+1-y), & \text{if } x > 0.5 \text{ and } y > 0.5. \end{cases}$
- $I_1 = I_2 = I_3 = 5\text{id}$ and $L = \text{id}$.
- $C_1 = C_2 = \begin{cases} \omega, & \text{if } \omega \leq 0.25, \\ \omega + 0.5(\omega - 0.25), & \text{if } \omega \geq 0.25, \end{cases}$ and

  $C_3 = \begin{cases} \omega, & \text{if } \omega \leq 0.25, \\ \omega + 0.25(\omega - 0.25), & \text{if } \omega \geq 0.25. \end{cases}$

Test experiments

Solution obtained with the GRASP algorithm.



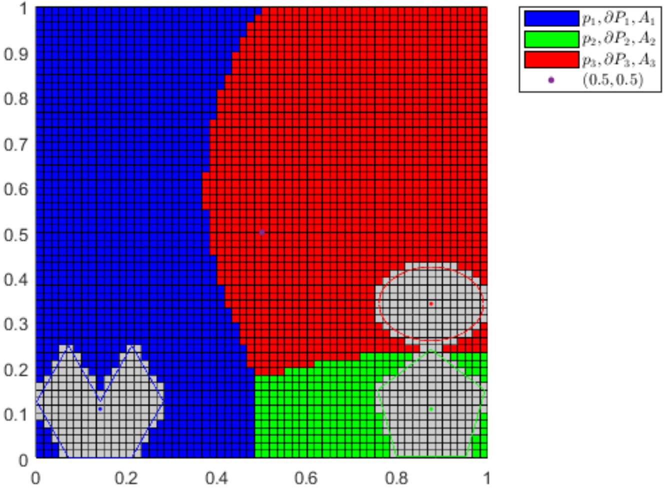
Test experiments

Performance of the GRASP algorithm:

- CPU time (preprocess+execution): 14333s+721s $\simeq$ 4.18h.
- Best objective value found: 1.2760.

Test experiments

Solution obtained with the MILP formulation.



Test experiments

Performance of the GRASP algorithm:

- CPU time (preprocess+execution): 14333s+721s $\simeq$ 4.18h.
- Best objective value found: 1.2760.

Performance of the MILP formulation (4h):

- CPU time (preprocess+execution): 3728s + 14400s $\simeq$ 5.04h.
- Best objective value found: 1.2760.
- GAP (%): 21.23.

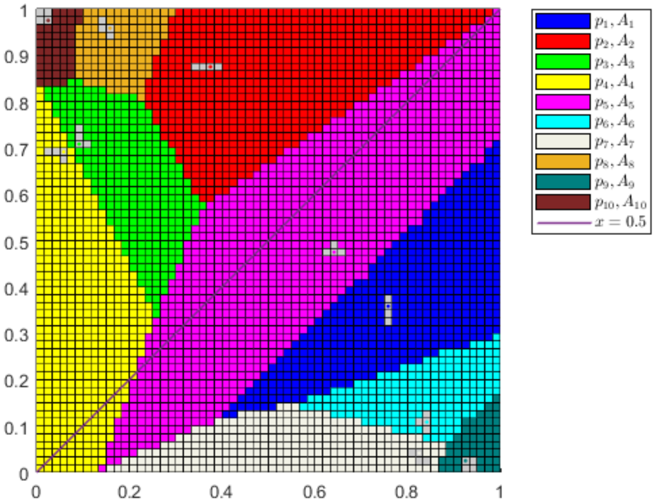
Test experiments

Test experiment 3:

- Dimensional facilities: ten tetrominoes.
- Utility: ℓ_2-distance.
- Regular grid with 60×60 cells.
- $D(q) = \begin{cases} 3(x-y), & \text{if } x \geq y, \\ 3(y-x), & \text{if } x < y. \end{cases}$
- $B(q) = 1.$
- $I_1 = I_2 = I_3 = 0,\ L = 0,$
  $C_i = \begin{cases} \omega, & \text{if } \omega \leq h_i cp, \\ h_i cp + 100(\omega - h_i)\omega, & \text{if } \omega \geq h_i cp, \end{cases}$, being $cp = 1/30$,
  $(h_1, h_2, h_3, h_4, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}) =$
  $(5, 5, 3, 3, 3, 3, 3, 3, 1, 1).$

Test experiments

Solution obtained with the GRASP algorithm.



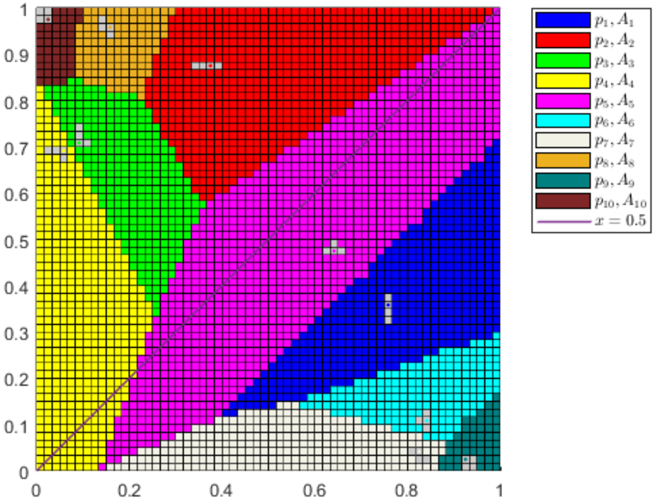
Test experiments

Performance of the GRASP algorithm:

- CPU time (preprocess+execution): 110915s+25303s $\simeq$ 37.84h.
- Best objective value found: 2.1139.

Test experiments

Solution obtained with the MILP formulation.



Test experiments

Performance of the GRASP algorithm:

- CPU time (preprocess+execution): 110915s+25303s $\simeq$ 37.84h.
- Best objective value found: 2.1139.

Performance of the MILP formulation (4h):

- CPU time (preprocess+execution): 4824s + 14400s $\simeq$ 5.34h.
- Best objective value found: 2.1139.
- GAP (%): 42.97.

Conclusions

Conclusions

- We give a first complete proof of existence of optimal solution of a general location-allocation problem with dimensional facilities.
- We also provide two methods (one is exact and the other one is heuristic) to solve this problem using sequences of solutions for a discrete approximation of the problem.
- Among other possible extensions, we would like to mention relaxing some conditions ensuring existence of optimal solutions, as for instance the continuity of the utilities in the objective function.
- In addition, these results can be extended to any finite dimension space at the price of increasing the complexity of the discrete models.

References

References

[1] Brazil, M., Ras, C. J., Thomas, D. A.: A geometric characterisation of the quadratic min-power centre. Eur. J. Oper. Res. **233**(1), 34-42 (2014)

[2] Carlier, G., Mallozzi, L.: Optimal monopoly pricing with congestion and random utility via partial mass transport. J. Math. Anal. Appl. **457**(2), 1218-1231 (2018)

[3] Feo, T. A., Resende, M. G.: Greedy randomized adaptive search procedures. J. Glob. Optim. **6**(2), 109-133 (1995)

[4] Fourer, R.: A simplex algorithm for piecewise-linear programming I: Derivation and proof. Math. Program. **33**(2), 204-233 (1985)

[5] Mallozzi, L., Passarelli di Napoli, A.: Optimal transport and a bilevel location–allocation problem. J. Glob. Optim. **67**(1-2), 207-221 (2017)

[6] Mallozzi, L., Puerto, J.: The geometry of optimal partitions in location problems. J. Optim. Lett. **12**(1), 203-220 (2018)

[7] Mallozzi, L., Puerto, J., Rodríguez-Madrena, M.: On location-allocation problems for dimensional facilities. J. Optim. Theory Appl. (2019) https://doi.org/10.1007/s10957-018-01470-y

Many thanks for your attention.

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