The Piecewise Linear Network Flow Problem

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Outline

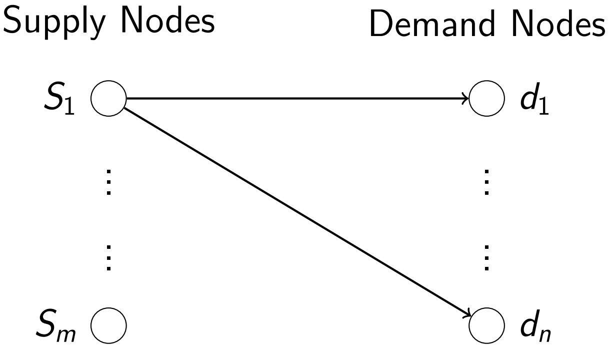
Outline

Problem description
- Network flow problems
- Piecewise linear costs
- Applications

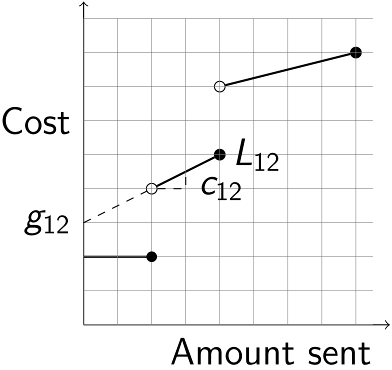
Models and strength of their LP

Exact method

Basic Flow (Transportation) Problem



Piecewise Linear Costs



- Also known as a ‘Staircase Cost‘ function if non-decreasing.
- Generalizes the Fixed-Charge Transportation Problem

Piecewise Linear Costs

Main applications

Piecewise Linear Costs

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- Different shipping modes: *Small packages, Less-than-Truckload, Truckloads*

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- Price discounts either all-unit or incremental, as often found in procurement

Piecewise Linear Costs

Main applications

- Different shipping modes: *Small packages, Less-than-Truckload, Truckloads*
- Price discounts either all-unit or incremental, as often found in procurement
- Linearize otherwise intractable functions

Outline

The Multiple-Choice Formulation

Model each mode l, between supplier i and customer j, by a binary variable v_{ijl} and the flow by x_{ijl}. Denoted by MCM.

The Multiple-Choice Formulation

$$
\begin{aligned}
\text{(MCF)} \quad & \min \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{l=1}^{q} (c_{ijl} x_{ijl} + g_{ijl} v_{ijl}), & \\
\text{s.t.} \quad & \sum_{i=1}^{n} \sum_{l=1}^{q} x_{ijl} = d_j, & \forall j, \\
& \sum_{l=1}^{q} v_{ijl} \le 1, & \forall (i,j), \\
& \sum_{j=1}^{m} \sum_{l=1}^{q} x_{ijl} \le S_i, & \forall\, i, \\
& x_{ijl} \le L_{ijl} v_{ijl}, & \forall (i,j,l), \\
& x_{ijl} \ge L_{ij,l-1} v_{ijl}, & \forall (i,j,l), \\
& x_{ijl} \ge 0, & \forall (i,j,l), \\
& v_{ijl} \in \{0,1\}, & \forall (i,j,l).
\end{aligned}
$$

The Multiple-Choice Formulation Con't

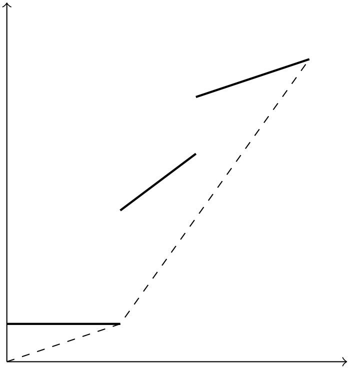
This is equivalent to the

- Convex Combination Model
- Incremental Model

w.r.t. the LP bound

The 'Textbook' Models

Provide a rather bad LP relaxation, i.e. the lower convex envelope.



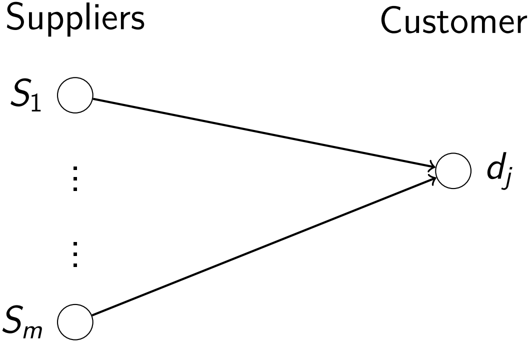
Two New Formulations

We present two new formulations based on a Dantzig-Wolfe reformulation while keeping in the master program either

- the capacity constraints (CBM)
- or the demand constraints (SBM)

The Customer-Based Model (CBM)

For each customer j pick one flow vector from the suppliers that satisfy his demand, such that the supply constraints are met. Each flow: a variable β_j^t.



The Customer-Based Model (CBM)

$$
\begin{aligned}
\text{(CBM)} \qquad & \min \sum_{j=1}^{m} \sum_{t \in T_j} \beta_j^t C_j^t & \\
& \text{s.t.} \ \sum_{t \in T_j} \beta_j^t = 1, & \forall j, \\
& \sum_{j=1}^{m} \sum_{t \in T_j} \beta_j^t x_{ij}^t \leq S_i, & \forall i, \\
& \beta_j^t \in \{0,1\}, & \forall j,\ \forall t \in T_j.
\end{aligned}
$$

The Pricing Problem - CBM

$$
\begin{aligned}
\text{(LP(CBM))} \qquad & \min \sum_{j=1}^{m} \sum_{t \in T_j} \beta_j^t C_j^t & & \\
& \text{s.t. } \sum_{t \in T_j} \beta_j^t = 1, & & \forall j, \pi_j \\
& \sum_{j=1}^{m} \sum_{t \in T_j} \beta_j^t x_{ij}^t \leq S_i, & & \forall i, \lambda_i \\
& \beta_j^t \geq 0, & & \forall j, \ \forall t \in T_j.
\end{aligned}
$$

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(\text{LP(CBM)}) \qquad & \min \sum_{j=1}^{m} \sum_{t \in T_j} \beta_j^t C_j^t \\
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& \beta_j^t \geq 0, && \forall j,\ \forall t \in T_j.
\end{aligned}
$$

A variable β_j^t prices out if

$$
C_j^t - \sum_{i=1}^{n} \lambda_i x_{ij}^t - \pi_j < 0
$$

The Pricing Problem - CBM

A variable β_j^t prices out if

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The Pricing Problem - CBM

$$
\begin{aligned}
(\mathrm{SSFCMCTP}_j^{\mathrm{CBM}}) \quad & -\pi_j + \min \sum_{i=1}^{n} \sum_{l=1}^{q} ((c_{ijl} - \lambda_j) x_{ijl} + g_{ijl} v_{ijl}), && \\
\text{s.t.} \quad & \sum_{i=1}^{n} \sum_{l=1}^{q} x_{ijl} = d_j, && \\
& \sum_{l=1}^{q} v_{ijl} \leq 1, && \forall i, \\
& x_{ijl} \leq L_{ijl} v_{ijl}, && \forall (i,l), \\
& x_{ijl} \geq L_{ijl-1} v_{ijl}, && \forall (i,l), \\
& x_{ijl} \geq 0, && \forall (i,l), \\
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\end{aligned}
$$

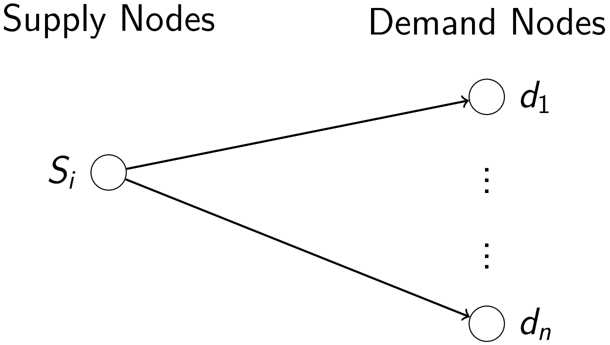
The Pricing Problem - CBM

The Pricing Problem is a Single-Sink, Fixed-Charge, Multiple-Choice Transportation Problem.

Christensen, Andersen and Klose (2012) give an efficient dynamic programming algorithm to solve this.

The Supplier-Based Model (SBM)

For each supplier i pick one flow vector to the customers that satisfies all demand constraints. All flows satisfy the capacity constraint. Each flow: a variable β_i^t.



The Supplier-Based Model (SBM)

$$
\begin{aligned}
\text{(SBM)} \quad \min \; & \sum_{i=1}^{n} \sum_{t \in T_i} \beta_i^t C_i^t, & \\
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& \beta_i^t \in \{0,1\}, & \forall\, i,\ \forall\, t \in T_i.
\end{aligned}
$$

The Pricing Problem - SBM

A variable β_i^t prices out if

$$C_i^t - \sum_{j=1}^{m} \eta_j x_{ij}^t - \omega_i < 0$$

The Pricing Problem - SBM

$$
\begin{aligned}
(\text{SSFCMCTP}_i^{\text{SBM}}) \quad & -\omega_i + \min \sum_{j=1}^{m} \sum_{l=1}^{q} ((c_{ijl} - \eta_j) x_{ijl} + g_{ijl} v_{ijl}), \\
\text{s.t.} \quad & \sum_{j=1}^{m} \sum_{l=1}^{q} x_{ijl} \le S_i, \\
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The Pricing Problem - SBM

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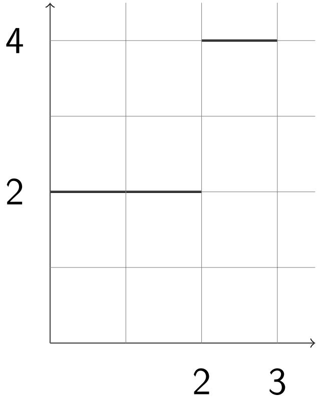
The CBM, Example

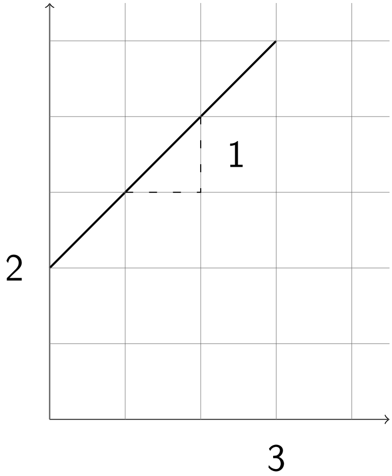
Ex. Cont.

Cost function at supplier 1 to either customer



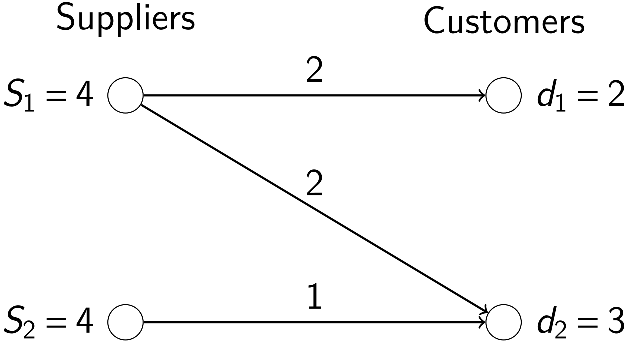
Ex. Cont.

Cost function at supplier 2 to either customer



Ex. Cont.

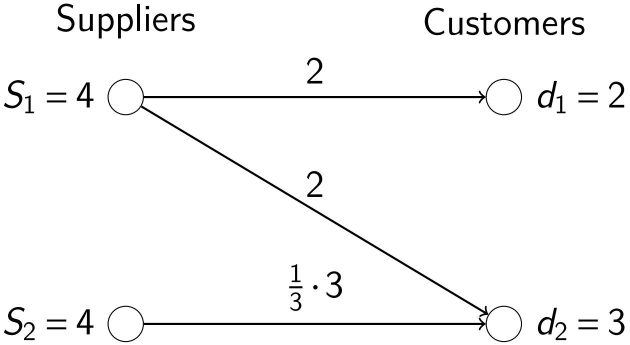
Optimal solution is



with a costs of $2+2+2+1 = 7$.

Ex. Cont.

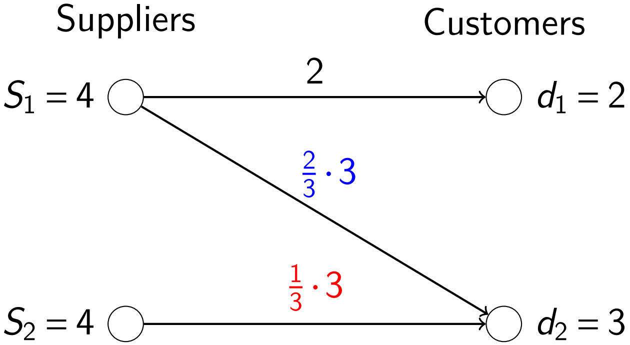
LP relaxation of the MCM



with a total cost of $2 + 2 + \frac{1}{3} \cdot (2 + 3 \cdot 1) = 5.\overline{66}$.

Ex. Cont.

LP relaxation of the CBM



with a costs of $2 + \frac{2}{3} \cdot 4 + \frac{1}{3} \cdot (2 + 3 \cdot 1) = 6.\overline{33}$.

Strength of the CBM

Easy to show it that $LP^*(MCM) \leq LP^*(CBM)$ and $LP^*(MCM) \leq LP^*(SBM)$.

Table: Average Gap Between the Optimal Solution and the Root Node LP over 5 instances

n	m	MCM	CBM	SBM
5	15	7.85%	0.53%	7.37%
	30	7.99%	0.06%	7.93%
	50	7.28%	0.03%	7.25%

Outline

Problem description

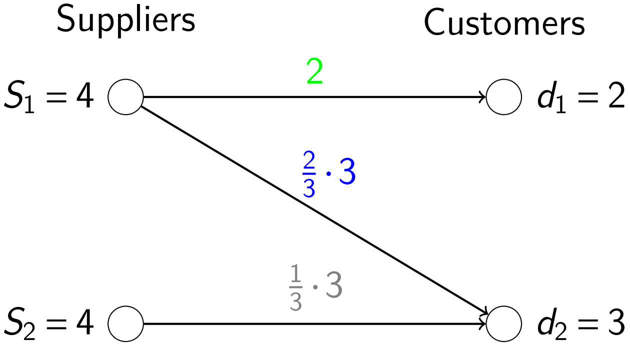
Models and strength of their LP

Exact method

- Valid inequality
- Branching
- Some results

Recall the Example

Take a good look at the solution from the previous example.



Cover Inequalities

A set of variables is a cover, $\mathscr{C}$, if

$$\sum_{(j,t)\in\mathscr{C}} x_{ij}^t > S_i$$

which gives rise to the (possibly) violated *generalized upper bound* constraint

$$\sum_{(j,t)\in\mathscr{C}} \beta_j^t \leq |\mathscr{C}| - 1$$

Valid Inequality for the Example

The optimal LP flow to customer 1 of $\begin{pmatrix} 2 & 0 \end{pmatrix}$ with $\beta_1^1 = 1$ and to customer 2 of $\begin{pmatrix} 3 & 0 \end{pmatrix}$ with $\beta_2^1 = 2/3$ can be cutoff as

$$2 + 3 > 4 \quad \text{and} \quad 1 + (2/3) > |\mathscr{C}| - 1 = 1$$

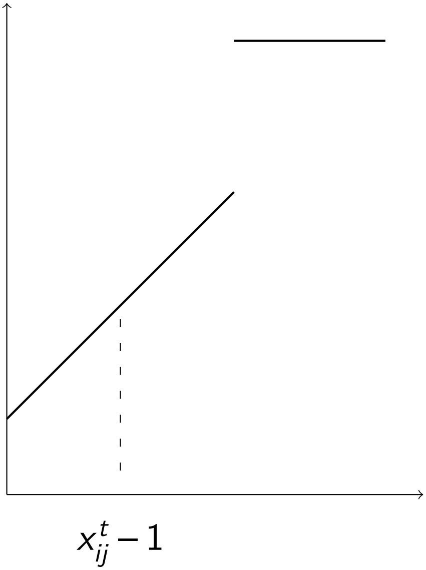
by adding the inequality

$$\beta_1^1 + \beta_2^1 \leq 1.$$

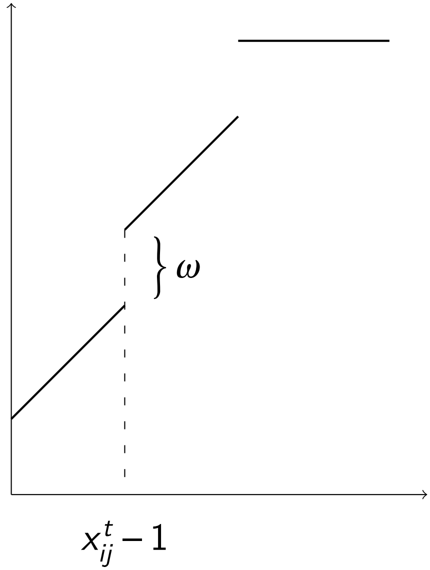
Interpretation in the Pricing Problem

- In the pricing problem, a cover inequality (from the Master) imply an extra penalty given by its associated dual variable ω.
- The cover, containing β_j^t can be extended by incorporating all variables β_j^k whose flow $x_{ij}^k \geq x_{ij}^t$.

Interpretation, cont.



Interpretation, cont.



Separation Problem

The separation problem is a multiple choice knapsack problem. It is solved by a simple heuristic or by an exact method.

Furthermore we strengthen the inequalities by lifting.

Fractional Variables

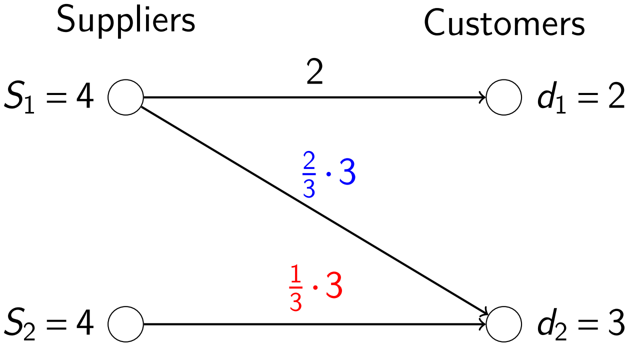
Obtain the solution in terms of the original variables

$$\overline{v}_{ijl} = \sum_{t \in \mathscr{S}} \beta_j^t$$

where $\mathscr{S}$ is the set of positive variables at j that use mode (i, j, l).

Example, Branching

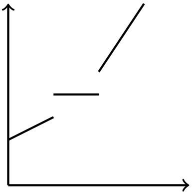
In the LP solution of the CBM we have $\overline{v}_{122} = \frac{2}{3}$ and $\overline{v}_{221} = \frac{1}{3}$.



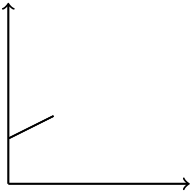
How to Choose a Variable

From a set of fractional variables choose the one that will invalidate most columns. If $\overline{v}_{ijl} \leq 0.5$ this is the number of columns when we fix $v_{ijl} = 1$. Otherwise it is the number of columns when we assume $v_{ijl} = 0$.

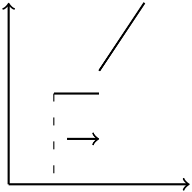
How to Branch



$\overline{v}_{ij2}$ is fractional



The to-zero node



The to-one node

Results

Tests compare our "customized" CBM to the MCM solved with CPLEX 12.2. It shows a reduction in runtime between 14% to 93%.

On the largest instances with 15 suppliers, 50 customers and 5 modes it is 75%!

Results

Table: Computational results for instances with 5 suppliers

		CPLEX	CBM					
			w/o. cuts			w. cuts		
m	q	secs.	secs.	#nodes	gap (%)	secs.	#nodes	gap (%)
15	2	0.17	**0.11**	62.2	0.61	0.15	30.6	0.52
	3	0.27	0.13	71.0	0.52	**0.09**	21.8	0.38
	4	0.49	**0.13**	123.4	0.31	0.14	31.8	0.26
	5	0.96	0.45	207.0	0.53	**0.35**	83.8	0.40
30	2	0.21	**0.09**	27.8	0.14	0.15	24.2	0.12
	3	0.61	**0.16**	45.4	0.06	0.19	29.0	0.12
	4	1.21	**0.21**	58.6	0.11	0.34	49.8	0.10
	5	1.21	**0.17**	43.8	0.06	**0.17**	23.4	0.04
50	2	0.37	0.08	15.4	0.03	**0.09**	8.6	0.02
	3	0.92	**0.27**	54.2	0.03	0.37	37.8	0.03
	4	3.44	**0.39**	66.6	0.02	**0.39**	39.4	0.02
	5	4.23	**0.27**	43.0	0.03	0.28	26.2	0.02

Results

Table: Computational results for instances with 10 suppliers

		CPLEX	CBM					
			w/o. cuts			w. cuts		
m	q	secs.	secs.	#nodes	gap (%)	secs.	#nodes	gap (%)
15	2	5.56	20.0	5432.6	3.42	**3.57**	306.6	2.33
	3	3.45	6.28	1813.8	1.72	**2.26**	207.8	1.05
	4	7.80	9.92	2654.6	1.74	**3.14**	344.6	1.18
	5	18.35	26.11	5271.0	1.45	**8.35**	790.6	0.99
30	2	1.38	1.64	382.2	0.56	**1.08**	85.4	0.43
	3	5.33	2.58	486.2	0.40	**2.56**	184.2	0.28
	4	14.40	9.24	1405.4	0.23	**7.79**	478.2	0.22
	5	15.34	4.39	659.4	0.33	**4.19**	288.6	0.28
50	2	3.55	2.17	273.8	0.18	**2.02**	113.8	0.14
	3	7.80	3.72	414.6	0.13	**4.67**	249.8	0.12
	4	16.83	**7.78**	789.0	0.12	8.49	419.8	0.12
	5	155.63	**101.26**	7947.4	0.16	104.99	2964.6	0.14

Results

Table: Computational results for instances with 15 suppliers

		CPLEX	CBM					
			w/o. cuts			w. cuts		
m	q	secs.	secs.	#nodes	gap (%)	secs.	#nodes	gap (%)
15	2	69.02	750.28 (1)	45409.6	4.20	**19.25**	495.8	0.89
	3	77.88	825.66 (1)	53146.4	3.91	**13.52**	252.6	1.0
	4	29.85	900.56 (1)	53784.2	2.16	**19.38**	321.0	0.68
	5	466.96	2881.59 (4)	134667.0	2.72	**102.45**	1713.8	0.89
30	2	90.78	838.06 (1)	49295.0	1.37	**62.39**	1519.4	0.59
	3	18.67	121.60	11527.8	0.65	**13.35**	245.4	0.22
	4	118.31	654.72	38326.6	0.67	**27.91**	681.8	0.28
	5	513.04	2130.62 (2)	84901.2	0.68	**251.58**	5007.4	0.31
50	2	33.47	353.33	25213.0	0.48	**23.30**	628.6	0.24
	3	136.46	411.01	23457.8	0.35	**105.89**	2289.8	0.17
	4	157.52	495.24	25577.4	0.27	**59.03**	1725.0	0.13
	5	690.88	369.74	20129.4	0.26	**168.72**	4391.8	0.15

The End!

Any ideas/comments are always welcome!