Conditions to LP relax the allocation variables of the Reliability Fixed-Charge Location Problem

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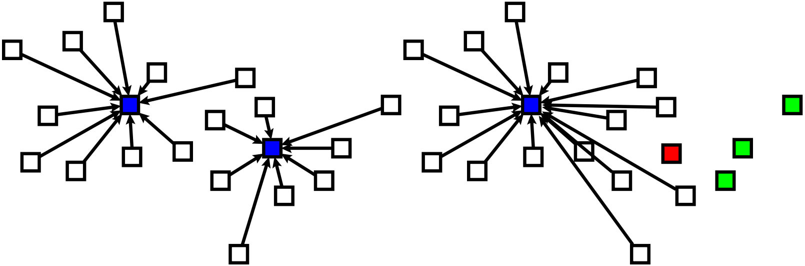
The problem RFLP

The Reliability Fixed-Charge Location Problem is a location problem where some facilities may fail and become unavailable with homogeneous probability, and these failures occur independently. For each customer, a sequence of assignments to opened facilities is defined and, at each scenario, the customer is served from the first facility in the sequence that has not failed.



Notation

- I is the set of customers,
- J is the set of potential facility locations
- NF is the set of candidate facilities that may not fail
- F is the set of candidate facilities that may fail
- h_i is the demand of customer i
- d_{ij} is the cost per unit of demand to serve customer i from facility j
- θ_i is the cost of not serving customer i
- q is the probability of each facility in F of failing
- $x_j \in \{0, 1\}$ and $Y_{ijr} \in \{0, 1\}$ are the decision variables

The model RFLP

The Reliability Fixed-Charge Location Problem (RFLP) was formulated by Snyder and Daskin as follows:

$$
\begin{aligned}
\text{(RFLP) min} \quad & \alpha\Big[\sum_{j\in J} f_j X_j + \sum_{i\in I}\sum_{j\in J} h_i d_{ij} Y_{ij0}\Big] \\
& + (1-\alpha)\sum_{i\in I} h_i\Big[\sum_{j\in NF}\sum_{r\in R} d_{ij} q^r Y_{ijr} + \sum_{j\in F}\sum_{r\in R} d_{ij} q^r (1-q) Y_{ijr}\Big] \\
s.t. \quad & X_u = 1 \\
& \sum_{j\in J} Y_{ijr} + \sum_{j\in NF}\sum_{s=0}^{r} Y_{ijs} = 1 && i\in I, r\in R \\
& \sum_{r\in R} Y_{ijr} \le X_j && i\in I, j\in J \\
& X_j \in \{0,1\} && j\in J \\
& Y_{ijr} \in \{0,1\} && i\in I, j\in J, r\in R
\end{aligned}
$$

where $R = \{0,\ldots,|F|-1\}$ and α is a value in $[0,1]$.

Targets

Snyder et al.assumed that $Y_{ijr} \in \{0, 1\}$, named as the allocation variables, can be relaxed. The goals of this work are focussed on this assumption, and they are to show:

- a counterexample to prove that the relaxation of the allocation variables is wrong,
- which allocation variables of the RFLP can be considered real in [0, 1] keeping the set of optimal binary solutions,
- a family of valid inequalities that forces the allocation variables to be integers,
- the conditions under which this assumption about the relaxation of allocation variables is correct.

Counterexample

The following example shows that to relax the allocation variables could be wrong:
Let $I = \{1\}$, $J = \{1, 2\}$, $NF = \{1\}$, $F = \{2\}$, $d_{11} = 1000$, $d_{12} = 1000$, $f_1 = f_2 = 0$, $h_1 = 1$, $\alpha = 0.5$ and $q = 0.1$. The optimal solution is $X_1^* = 1$, $X_2^* = 0$ and

$$(Y^*_{1jr})_{(j\in J)\times(r\in R)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

with objective value: $v^* = 1000$
However, if all the allocation variables are relaxed the optimal solution of the relaxed problem is better than (X^*, Y^*). Indeed, the solution given by $X_1 = X_2 = 1$ and

$$(Y_{1jr})_{(j\in J)\times(r\in R)} = \begin{pmatrix} 0.50 & 0 \\ 0.50 & 0.50 \end{pmatrix}$$

is feasible for the relaxed problem and its objective value v is smaller than v^*, $v = 997.50$

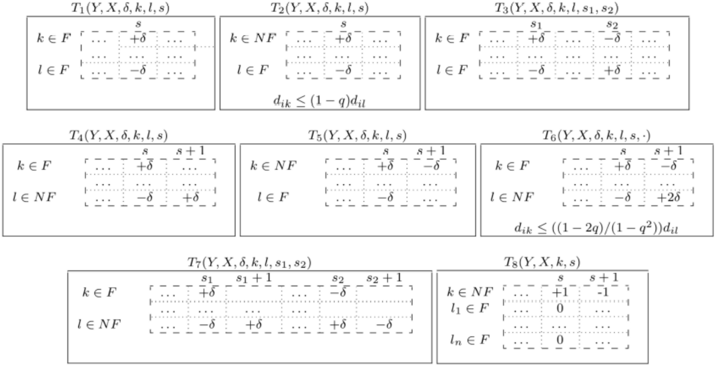
On relaxing the allocation variables

The conclusions of our study are the following:

- allocation variables type NF can be relaxed,
- allocation variables type F can be relaxed,
- allocation variables type F and type NF can be relaxed simultaneously if $d_{ik} \leq (1-q)d_{il}$ and $d_{ik} \leq ((1-2q)/(1-q)^2)d_{il}$ for all $k \in F, l \in NF$ such that $d_{ik} \leq d_{il}$,
- allocation variables type F and type NF can be relaxed simultaneously if we add the family of constraints: $\sum_{j \in NF} \sum_{r \in R} Y_{ijr} = 1 \qquad \forall i \in I$

Transformations

To demonstrate these conclusions, we use different transformations that allow, from a solution to reach another point -not necessarily feasible- with better objective value. These transformations are:



On relaxing the allocation variables type NF

Since $Y_{ijr} \in \{0,1\}$ for all $i \in I, j \in F, r \in R$,
constraint $\sum_{j\in F} Y_{ijr} + \sum_{j\in NF} \sum_{s=0}^{r} Y_{ijs} = 1 \quad \forall i \in I, r \in R$
implies that $\sum_{j\in NF} \sum_{s=0}^{r} Y_{ijs} \in \{0,1\}$ for all $i \in I, r \in R$.

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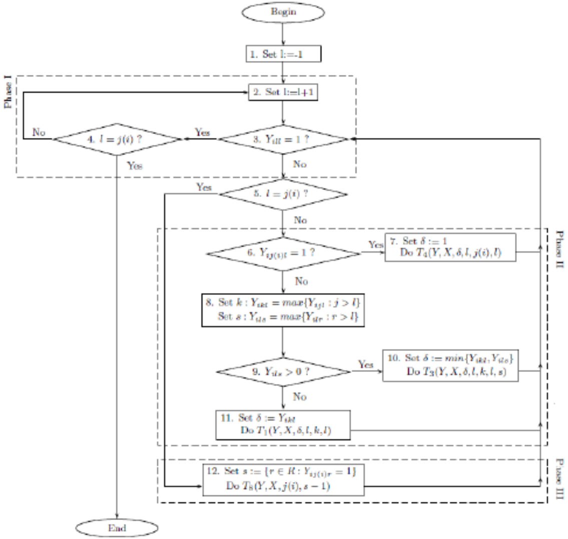
Then $\sum_{j\in NF} \sum_{s=0}^{r} Y_{ijs} - \sum_{j\in NF} \sum_{s=0}^{r-1} Y_{ijs} = \sum_{j\in NF} Y_{ijr} = Y_{ij(i)r} \in \{0,1\}$ for all $i \in I, r \in R$. holds. $\square$

Algorithms

We can not use any one transformation with any one value to improve the objective value given that the new point obtained could be not feasible.
For this reason, we have developed different algorithms to indicate what transformations and what values we can use to improve any one solution without leaving the feasible region, until the optimal solution is reached.

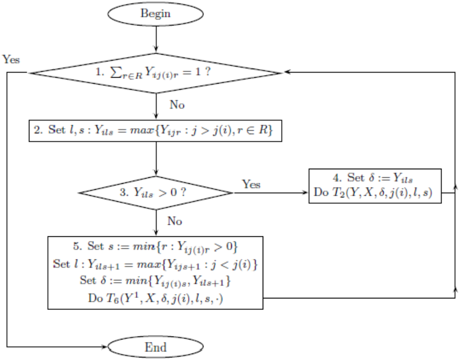
On relaxing the allocation variables type F



Conditions to relax simultaneously all allocation variables

The conditions to relax simultaneously all allocation variables are: $d_{ik} \leq (1-q)d_{il}$ and $d_{ik} \leq ((1-2q)/(1-q)^2)d_{il}$ for all $k \in F, l \in NF$ such that $d_{ik} \leq d_{il}$

We prove these conditions using the following algorithm:

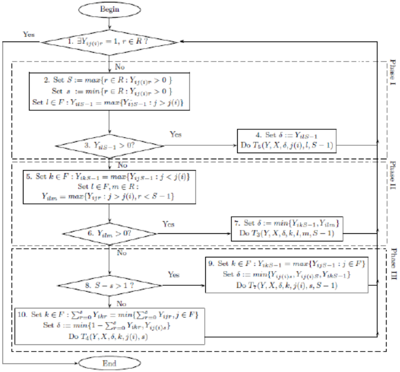


On relaxing simultaneously all allocation variables without conditions

We can relax all allocation variables without conditions but adding the family of constraints:

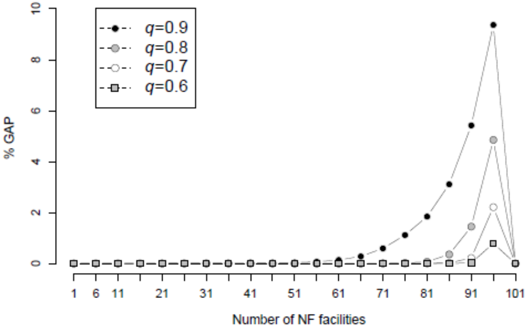
$\sum_{j \in NF} \sum_{r \in R} Y_{ijr} = 1 \qquad \forall i \in I$

We prove this using the following algorithm:



Error

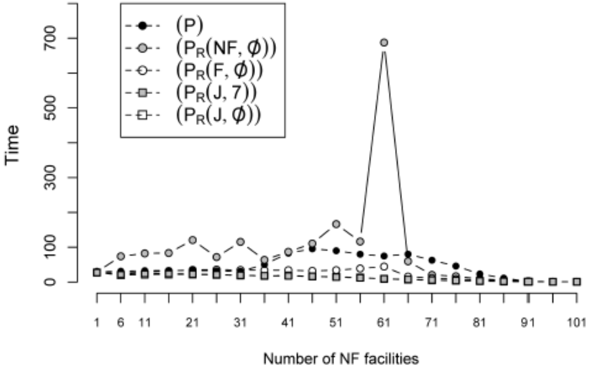
This figure shows the error detected by computational experience from the files gapA332, gapA1332 and gapA2332 in the web page `http://www.math.nsc.ru/AP/benchmarks/UFLP/Engl/uflp_dg_eng.html` for *alpha* $= 0.5$.



$GAP\% = 100(v^*(P) - v^*P_R)/v^*P_R).$

Processing time

This figure shows the processing time when we apply the differents methods to the files gapA332, gapA1332 and gapA2332 in the web page `http://www.math.nsc.ru/AP/benchmarks/UFLP/Engl/uflp_dg_eng.html` for $q = 0.9$



Algorithms computer tool

We have developed a computer tool that illustrates these algorithms.
This computer tool has been developed in html+javascript and it can be tested from the web page:

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http://gestionderecursosyoptimizacion.edu.umh.es/data-library/rflp-relaxation-variables/

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- *We can LP relax the allocation variables if the parameters of the problem accomplish:*
  $d_{ik} \leq (1-q)d_{il}$ *and* $d_{ik} \leq ((1-2q)/(1-q)^2)d_{il}$ *for all* $k \in F, l \in NF$ *such that* $d_{ik} \leq d_{il}$
- *If the parameters of the problem don't accomplish the previous condition, the most effective way to LP relax all allocation variables is to add the family of constraints:*
  $\sum_{j \in NF} \sum_{r \in R} Y_{ijr} = 1 \qquad \forall i \in I$

Thanks for your attention