Robust p-median problem with vector autoregressive demands

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28th November 2015

Classic p-median problem

$$\min_{\mathbf{x},\mathbf{y}} \quad \sum_{i=1}^{n} w_i \sum_{j=1}^{m} c_{ij} x_{ij}$$

$$\text{s.t} \quad \begin{cases} \sum_{j=1}^{m} x_{ij} = 1 & \forall j \\ x_{ij} \leq y_i & \forall i, j \\ \sum_{j=1}^{m} y_j = p & \\ y_j, x_{ij} \in \{0, 1\} & \end{cases} \qquad (p\text{-median})$$

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Uncertain in practice

Background

Robust location problems

Uncertainty over the demand:

- Distributional assumptions
- Scenario analysis

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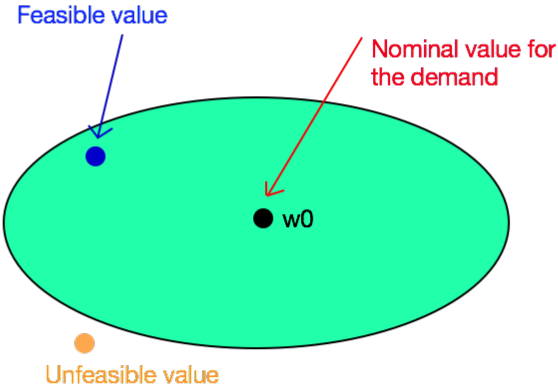
Construct uncertainty sets for the demand around nominal value

Uncertainty set around nominal value



Robust p-median problem

$$\min_{\mathbf{x},\mathbf{y}\in R} \max_{w\in U} \quad \mathbf{w}' F(\mathbf{x}) \qquad \text{(Robust } p\text{-median)}$$

- U: uncertainty set for the demand
- $F_i(\mathbf{x}) = \sum_{j=1}^{m} c_{ij} x_{ij}$: cost function
- R: feasible region of problem (p-median)

Disadvantages

Which statistical procedures are used to predict the demand?

- No control
- Distributional assumptions?

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Moreover:

Temporal correlation

Usually disregarded

Background

Baron, O., Milner, J., & Naseraldin, H. (2011). Facility location: A robust optimization approach. Production and Operations Management, 20(5), 772-785.

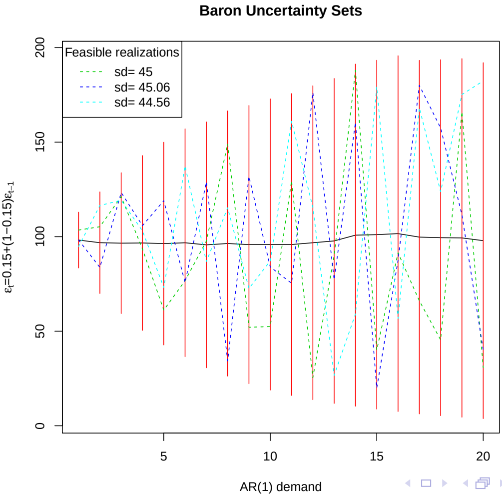
- Multi-period setting
- Uncertainty sets for the demands:
  - l_1-norm (box uncertainty)
  - l_2-norm (elliptical uncertainty)

Motivation

Disadvantages Baron et al. (2011)

- Given nominal values for the demand
- Demand realization for time t does not depend on the realization of the demand for time $t-1$

Example: Baron et al. (2011) uncertainty sets



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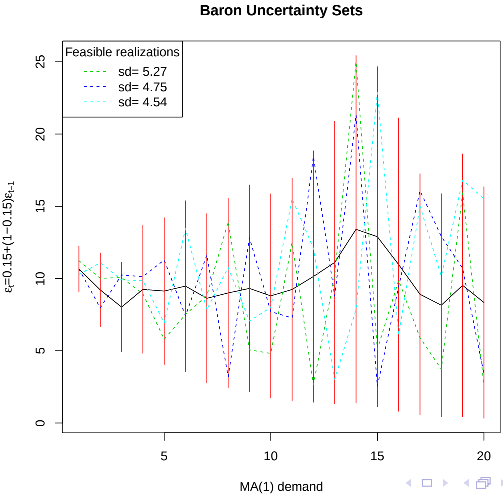

Motivation

Differences with Baron et al. (2011)

- Given nominal values for the demand $\rightarrow$ **VAR coefficients**
- Demand realization for time t does not depend on the realization of the demand for time $t-1$ $\rightarrow$ **realization of the demand must preserve inner behaviour**

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We have already used robustness+AR processes with very good results:

Carrizosa, E., Olivares-Nadal, A.V., & Ramírez-Cobo, P. Robust newsvendor problem with autoregressive demand. To appear in Computers & Operations Research.

Vector Autoregressive Models of order p (VAR(p))

Multivariate time series: correlation along time and between clients.

VAR(p)

$$\mathbf{w}_t = \boldsymbol{\alpha} + \sum_{k=1}^{p} A_k \mathbf{w}_{t-k} + \boldsymbol{\epsilon}_t$$

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- $\mathbf{w}_t = \left(w_t^1, \ldots, w_t^n\right)'$: the demands of all clients at time t, known up to time T,
- $A_1, \ldots, A_p \in \mathbb{R}^{n \times n}$, $\boldsymbol{\alpha} \in \mathbb{R}^n$: VAR parameters, given or estimated.
- $\boldsymbol{\epsilon}_t$: random term, shocks.

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Uncertainty sets

Demand following a VAR

$$U_{\tilde{\mathbf{w}}} = \{\tilde{\mathbf{w}} : \quad M\tilde{\mathbf{w}} - \tilde{\boldsymbol{\epsilon}} = \mathbf{b}, \quad \tilde{\mathbf{w}} \geq 0\}$$

Demand following a VAR

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M: known, contains regression coefficients

$$M = \begin{bmatrix} I_{n\times n} & 0 & \dots & & & 0 \\ -A_1 & I_{n\times n} & 0 & \dots & & 0 \\ -A_2 & -A_1 & I_{n\times n} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ -A_p & -A_{p-1} & -A_{p-2} & \dots & & \\ \vdots & & & & & \\ 0 & \dots & & & & I_{n\times n} \end{bmatrix}$$

Demand following a VAR

$$U_{\tilde{\mathbf{w}}} = \{\tilde{\mathbf{w}} : \quad M\tilde{\mathbf{w}} - \tilde{\boldsymbol{\epsilon}} = \mathbf{b}, \quad \tilde{\mathbf{w}} \geq 0\}$$

b: known, contains intercepts and historical demands

$$\mathbf{b} = \begin{bmatrix} \boldsymbol{\alpha} + \sum_{k=1}^{p} A_k \mathbf{w}_{T-k} \\ \boldsymbol{\alpha} + \sum_{k=2}^{p} A_k \mathbf{w}_{T+1-k} \\ \vdots \\ \boldsymbol{\alpha} + A_p \mathbf{w}_{T-1} \\ \boldsymbol{\alpha} \\ \vdots \\ \boldsymbol{\alpha} \end{bmatrix}$$

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Bounding the errors

$$U_{\tilde{\boldsymbol{\epsilon}}} = \{\tilde{\boldsymbol{\epsilon}} : \quad \|\tilde{\boldsymbol{\epsilon}}\| \leq \delta\}$$

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$$\min_{\mathbf{x},\mathbf{y} \in R} \quad \max_{\substack{\tilde{\mathbf{w}} \in U_{\tilde{\mathbf{w}}} \\ \tilde{\boldsymbol{\epsilon}} \in U_{\tilde{\boldsymbol{\epsilon}}}}} \quad \tilde{\mathbf{w}}\mathbb{F}(\mathbf{x})$$

Theorem

The robust p-median problem

$$\min_{\mathbf{x},\mathbf{y}\in R} \max_{\substack{\tilde{\mathbf{w}}\in U_{\tilde{\mathbf{w}}}\\ \tilde{\boldsymbol{\epsilon}}\in U_{\tilde{\boldsymbol{\epsilon}}}}} \tilde{\mathbf{w}}\mathbb{F}(\mathbf{x})$$

is equivalent to the following optimization problem:

$$\min_{\substack{\mathbf{x},\mathbf{y}\in R\\ \boldsymbol{\lambda}\geq 0}} \mathbf{b}' G(\mathbf{x},\boldsymbol{\lambda}) + \delta\| G(\mathbf{x},\boldsymbol{\lambda})\|^*$$

where $G(\mathbf{x},\boldsymbol{\lambda}) = (M^{-1})'(\mathbb{F}(\mathbf{x}) + \boldsymbol{\lambda})$ *and* $\boldsymbol{\lambda} \in \mathbb{R}^{nh}$.

Advantages of the new formulation

- Gets rid of the minmax formulation: now only minimize
- Convex objective function: linear term plus a regularization term
- Uncertainty sets disappear

Sensitivity Analysis

Corollary

Let $(\mathbf{x}^0, \mathbf{y}^0)$ *be the solution of the deterministic p-median problem* (p-median). *Then, the maximum* δ *such that the solution to the robust p-median problem* (1) *is still* $(\mathbf{x}^0, \mathbf{y}^0)$ *is:*

$$\delta^0 = \min_{\substack{\mathbf{x},\mathbf{y}\in R \\ \boldsymbol{\lambda}\geq 0}} \quad \frac{\hat{\mathbf{w}}'(\mathbb{F}(\mathbf{x}) - \mathbb{F}(\mathbf{x}^0))}{\|(M^{-1})'(\mathbb{F}(\mathbf{x}^0) + \boldsymbol{\lambda})\|^* - \|(M^{-1})'(\mathbb{F}(\mathbf{x}) + \boldsymbol{\lambda})\|^*},$$

where $\hat{\mathbf{w}} = M^{-1}\mathbf{b}$ *is the estimation of the demand via the VAR model.*

Remark

All results can be extended to Uncapacitated Facility Location Problem (UFLP)

Preliminary results!!!

This is a working paper, in development

Future work

- Empirically test the performance of our approach
- Compare against Baron et al. (2011)

Thank you for your attention!

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