On the Hierarchical Rural Postman Problem on a mixed graph

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Outline

Given:

- $G = (V, E, A)$, a strongly connected mixed graph
- $V = \{1, ..., n\}$, the set of vertices
- E, the set of edges
- A, the set of arcs
- $E_R = E_R^1 \cup E_R^2 \cup ... \cup E_R^p$, the set of required edges to serve in a hierarchical order
- $A_R = A_R^1 \cup A_R^2 \cup ... \cup A_R^p$, the set of required arcs to serve in a hierarchical order
- a depot (vertex 1) and a single vehicle

HIERARCHICAL ORDER: all the edges and arcs in $E_R^k \cup A_R^k$ must be serviced before those in $E_R^m \cup A_R^m$ if $k < m$.

Costs:

- $\hat{c}_{ij}$: the cost of traversing a link (i, j) that has not been serviced yet
- $\bar{c}_{ij}$: the cost of traversing and servicing link (i, j)
- $\tilde{c}_{ij}$: the cost of traversing a link (i, j) that has already been serviced
- c_{ij} associated with each non-required link $(i, j) \in E_{NR} \cup A_{NR}$ representing the cost of traversing it ($E_{NR} = E \setminus E_R$ and $A_{NR} = A \setminus A_R$)

The cost of traversing an edge is the same in both directions.

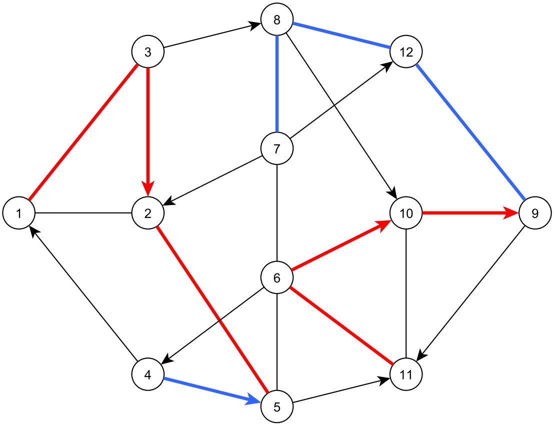
- OBJECTIVE: to find a minimum cost tour starting and ending at the depot and servicing all the required links in the hierarchical order.

Note:

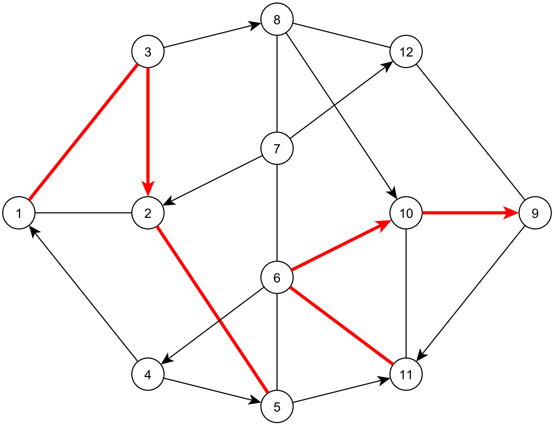
- a tour for the HMRPP $\rightarrow$ a series of consecutive connected paths, starting from a “starting node” and ending in an “ending node”
- the subgraph induced by the required links $G(E_R \cup A_R)$ and the subgraphs $G(E_R^k \cup A_R^k)$, for all k, do not need to be connected

Graphical representation

Required links in the first hierarchy Required links in the second hierarchy

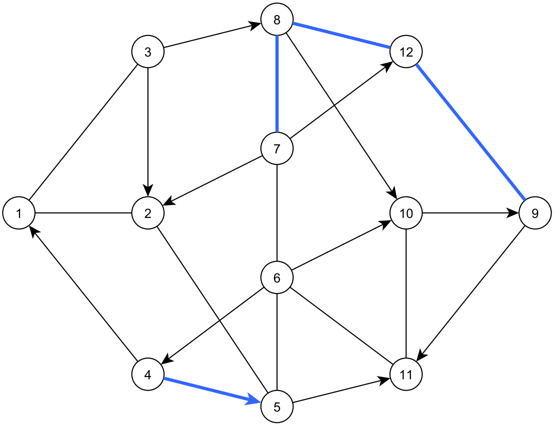


Required links in the first hierarchy

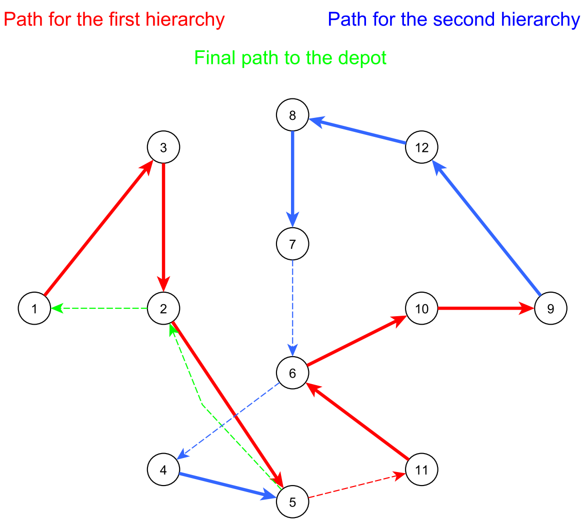


Graphical representation - Second Hierarchy

Required links in the second hierarchy



Graphical representation - Feasible Solution



Literature Review

Hierarchical Chinese Postman Problem:

- Postman Tour on a Graph with Precedence Relation on Arcs (Dror, Stern, Trudeau, 1987).
- Postman routing problem in a hierarchical network (Alfa, Liu, 1988).
- An algorithm for the hierarchical Chinese postman problem (Ghiani, Improta, 2000).
- Solving the hierarchical Chinese postman problem as a rural postman problem (Cabral, Gendreau, Ghiani, Laporte, 2004).
- On the Hierarchical Chinese Postman Problem with linear ordered classes (Korteweg, Volgenant, 2006).
- Vehicle Routing for Urban Snow Plowing Operations (Perrier, Langevin, Amaya, 2008).

Hierarchical Mixed Rural Postman Problem:

- The Hierarchical Mixed Rural Postman Problem (Colombi, Corberán, Mansini, Plana, Sanchis, Submitted).
- The Hierarchical Mixed Rural Postman Problem: Polyhedral analysis and a branch-and-cut algorithm (Colombi, Corberán, Mansini, Plana, Sanchis, Submitted).

Problem formulation

We define the following variables and consider a fictitious level $p+1$ associated with the trip back to the depot:

- x^k_{ij}, $k = 1, \ldots, p, p+1$, denoting the number of times arc (i, j) is traversed in level k,
- x^k_{ij} and x^k_{ji}, $k = 1, \ldots, p, p+1$, denoting the number of times edge $\{i, j\}$ is traversed from i to j and from j to i in level k, and
- z^k_i, $i \in V^k_F$, $k = 1, \ldots, p$, which takes value 1 if the path servicing the links in level k ends at node i (and therefore the path corresponding to hierarchical level $k+1$ begins at i), and zero otherwise.

Problem formulation

$$Minimize \quad \sum_{k=1}^{p+1} \Big(\sum_{\{i,j\} \in E} c_{ij}^k (x_{ij}^k + x_{ji}^k) + \sum_{(i,j) \in A} c_{ij}^k x_{ij}^k \Big) +$$

$$+ \sum_{\{i,j\} \in E_R} (\bar{c}_{ij} - \tilde{c}_{ij}) + \sum_{(i,j) \in A_R} (\bar{c}_{ij} - \tilde{c}_{ij})$$

where

$$c_{ij}^k = \begin{cases} \hat{c}_{ij}, & \text{if } (i,j) \in E_R^s \cup A_R^s, s > k; \\ \tilde{c}_{ij}, & \text{if } (i,j) \in E_R^s \cup A_R^s, s \leq k; \\ c_{ij}, & \text{if } (i,j) \in E_{NR} \cup A_{NR}. \end{cases}$$

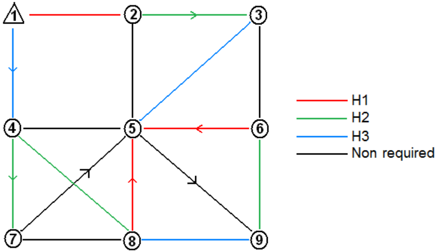
and remember that

- $\hat{c}_{ij}$ is the cost of traversing a link (i,j) that has not been serviced yet
- $\bar{c}_{ij}$ is the cost of traversing and servicing link (i,j)
- $\tilde{c}_{ij}$ is the cost of traversing a link (i,j) that has already been serviced
- c_{ij} is the cost of traversing a non-required link (i,j).

Problem formulation

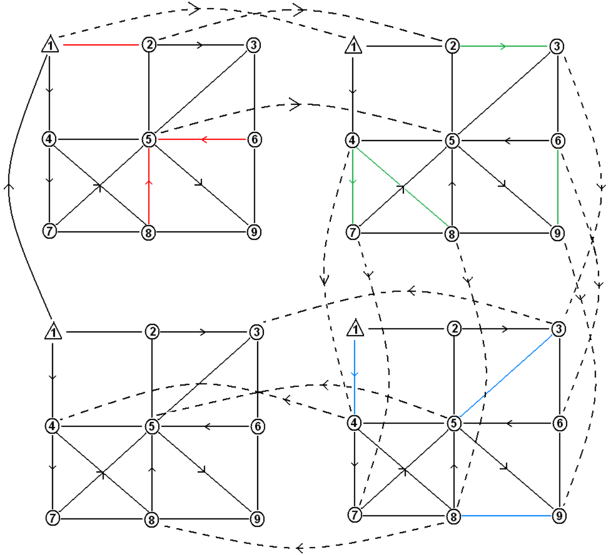
$$
\begin{aligned}
x_{ij}^k + x_{ji}^k &\geq 1, & &\forall\ \{i,j\} \in E_R^k, k = 1, ..., p & (1)\\
x_{ij}^k &\geq 1, & &\forall\ (i,j) \in A_R^k, k = 1, ..., p & (2)\\
x^k(\delta^-(i)) + z_i^{k-1} &= x^k(\delta^+(i)) + z_i^k, & &\forall\ V, k = 1, ..., p+1 & (3)\\
x^k(\delta^-(S)) + z^{k-1}(S) &\geq 1, & &S = (\cup_{i \in Q} R_i^k) \cup W, & (4)\\
& & &Q \subseteq \{1, ..., m_k\}, &\\
& & &W \subseteq V \setminus V_R^k, k = 1, ..., p &\\
z^k(V_R^k) &= 1, & &\forall\ k = 0, 1, ..., p+1 & (5)\\
z_1^0 = z_1^{p+1} &= 1 & & & (6)\\
x_{ij}^k, x_{ji}^k &\geq 0 \text{ and integer}, & &\forall\ \{i,j\} \in E, k = 1, ..., p+1 & (7)\\
x_{ij}^k &\geq 0 \text{ and integer}, & &\forall\ (i,j) \in A, k = 1, ..., p+1 & (8)\\
z_i^k &\in \{0,1\} & &\forall\ i \in V_R^k, k = 1, ..., p & (9)
\end{aligned}
$$

HMRPP polyhedron



Let us describe the relationship between the HMRPP on G and the MRPP defined on an extended graph $\overline{G}$ consisting of $p+1$ copies of graph G: $G^1, \ldots, G^{p+1}$, where for each $k = 1, \ldots, p$ the links in the set $E_R^k \cup A_R^k$ are considered required whereas all the links in G^{p+1} are considered nonrequired.

HMRPP polyhedron



HMRPP polyhedron

Associated with each link in G^k, we consider the variable x_{ij}^k.
For each vertex $i \in V_F^k$ in G^k, we add an arc in $\overline{G}$ from this vertex to the copy of the same vertex i in graph G^{k+1}. With this arc, we associate the variable z_i^k, representing the number of times this arc is traversed.

The HMRPP on graph G can be seen as the problem of finding a minimum cost path on graph $\overline{G}$ traversing all the required links from vertex $1^{[1]}$ (vertex 1 in G^1) to vertex $1^{[p+1]}$ (vertex 1 in G^{p+1}).

If we add to $\overline{G}$ the arc $\left(1^{[p+1]}, 1^{[1]}\right)$ and consider it as required, the HMRPP on graph G can be seen as a MRPP on graph $\overline{G}$ with the additional condition that the required arc $\left(1^{[p+1]}, 1^{[1]}\right)$ is traversed exactly once.

HMRPP polyhedron

HMRPP(G) is the convex hull of the HMRPP tours (x, z), each one with $(p+1)(2|E|+|A|)+\sum_{k=1}^{p}|V_F^k|$ variables.

Since the HMRPP on G is equivalent to the MRPP on $\overline{G}$ with the condition $z_1^{p+1}=1$, and $z_1^{p+1}\geq 1$ is a facet of MRPP($\overline{G}$),

$$\text{dim}(\text{HMRPP}(G)) = \text{dim}(\text{MRPP}(\overline{G})) - 1$$

If $\overline{G}$ is strongly connected, the dimension of MRPP($\overline{G}$) is the number of variables minus the number of vertices in $\overline{G}$ plus one:

$$\text{dim}(\text{HMRPP}(G)) = (p+1)(2|E|+|A|)+\sum_{k=1}^{p}|V_F^k|-(p+1)|V|+1 =$$

$$= (p+1)\big(2|E|+|A|-|V|\big)+\sum_{k=1}^{p}|V_F^k|+1.$$

Theorem

$z_1^{p+1}\geq 1$ is facet inducing of the polyhedron MRPP($\overline{G}$).

HMRPP polyhedron

Polyhedron HMRPP(G) is hard to study. Then, we study a very similar polyhedron, HMRPP$^*(G)$, the convex hull of the vectors (x, z, z_1^{p+1}) associated with the MRPP tours on graph $\overline{G}$.

This polyhedron satisfies

$$\dim\big(\text{HMRPP}^*(G)\big) = \dim\big(\text{HMRPP}(G)\big) + 1.$$

The polyhedron HMRPP$^*(G)$ is exactly the polyhedron MRPP($\overline{G}$).

Note that $\overline{G}$ is a strongly connected mixed graph with a special topological structure: it is formed by $p + 1$ strongly connected subgraphs (those of the hierarchies) which are connected with arcs forming a cycle.

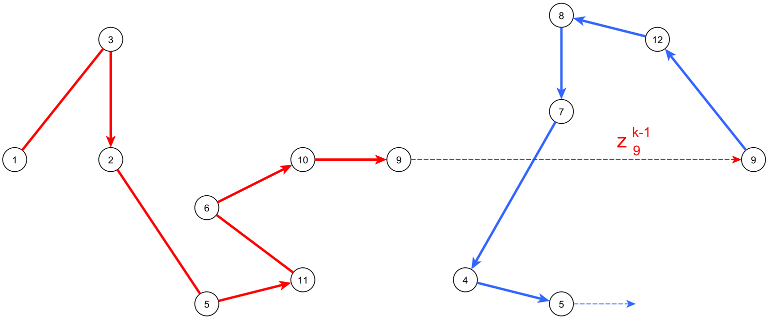
HMRPP polyhedron

Connectivity inequalities

$$x^k(\delta^-(S)) + z^{k-1}(S) \geq 1, \quad \begin{array}{l} S = (\cup_{i\in Q} R_i^k) \cup W, \\ Q \subseteq \{1, ..., m_k\}, \\ W \subseteq V \setminus V_R^k, k = 1, ..., p \end{array}$$

are facet inducing for HMRPP*(G) if subgraphs $G(S)$ and $G(V \setminus S)$ are strongly connected, $V_F^{k-1} \cap (V \setminus S) \neq \emptyset$, and $V_F^k \cap (V \setminus S) \neq \emptyset$.

HMRPP polyhedron



z_9^{k-1} keeps the information that the path of hierarchy $k-1$ ends in node 9 and the path of hierarchy k starts in node 9.

HMRPP polyhedron

Parity inequalities

$$x^k\big(\delta^+(S)\big) + x^k\big(\delta(S)^-\big) + z^{k-1}(S) + z^k(S) \geq \left|\delta_R^k(S)\right| + 1,$$
$$\forall S \subset V, \text{ such that } \left|\delta_R^k(S)\right| \text{ is odd}$$

are facet inducing for HMRPP$^*(G)$ if subgraphs $G(S)$ and $G(V \setminus S)$ are strongly connected, $V_F^{k-1} \cap (V \setminus S) \neq \emptyset$, and $V_F^k \cap (V \setminus S) \neq \emptyset$.

HMRPP polyhedron

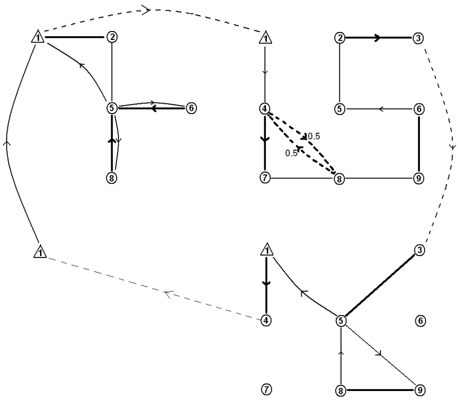
K-C inequalities

$$(K-2)\, x^k\,(M_0 : M_K) + \sum_{\substack{0 \le i < j \le K \\ \{i,j\} \neq \{0,K\} \\ i,j \neq L}} (j-i)\, x^k\,(M_i : M_j) +$$

$$\sum_{\substack{0 \le j \le K \\ j \neq L}} |L-j| \sum_{s \in M_j} (z_s^k + z_s^{k-1}) \ge 2(K-1) + (K-2)\left|(M_0 : M_K)_R^k\right|$$

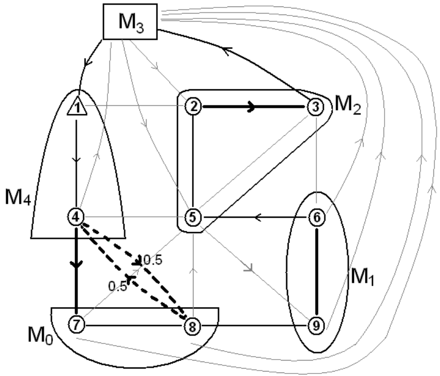
are facet inducing for HMRPP$^*(G)$ if several conditions hold.

HMRPP polyhedron



A fractional solution that can be cut by a K-C inequality

HMRPP polyhedron



A K-C configuration with $K = 4$ and $L = 3$ that cuts the fractional solution

Branch & Cut

The initial LP contains

- traversing inequalities (1) and (2)
- symmetry equations (3)
- a connectivity inequality (4) associated with each R-set in each hierarchy
- equations (6) and (7)

Separation algorithms:

- Heuristic separation algorithm for Connectivity inequalities
- Heuristic separation algorithm for Parity inequalities
- Heuristic separation algorithm K-C inequalities
- Exact separation algorithm for Connectivity inequalities (sometimes)

Branch & Cut

The cutting plane at the root node consists of applying

- the heuristic algorithm for separating connectivity ineq. with different values of parameter $\epsilon \in \{0, 0.25, 0.5\}$. Each value is used only if the previous ones fail to find violated inequalities,
- heuristic separation of parity inequalities, and
- K-C heuristics in a given hierarchy only when, for this hierarchy, the heuristic connectivity algorithm with $\epsilon = 0$ fails and no violated K-C inequalities have been found in previous hierarchies.

At the remaining nodes of the branch-and-cut tree, K-C separation is not applied.

Branch & Cut

We enable CPLEX to use all its classical cuts, the strong branching procedure is selected and the preference for branching is set on variables z_i^k. The best bound strategy is used to select unexplored nodes from the tree, and the solver is set to try to improve the best bound (MIP emphasis parameter). Basic heuristics of the solver are disabled.

For each instance, we use as cutoff value the best known value obtained by the Tabu Search of Colombi et al. (2015). Time limit has been set to 3600 seconds by using only one thread. All other parameters of the branch-and-cut are automatically selected by the solver.

Computational results: Instances

Set of instances (Colombi et al., 2015):

- **First group.** Small instances derived from the MGRP: ALBA_k, ALDA_k and MADR_k, with $k = 2, .., 5$ hierarchies.
- **Second group.** Large instances derived from the MRPP: RB_k and RD_k, $k = 2, .., 5$.

	#	# nodes	# links req	# links
ALBA	10	116	116 - 160	174 - 174
ALDA	10	214	157 - 300	351 - 351
MADR	10	196	215 - 288	316 - 316
RB	18	357 - 498	262 - 994	884 - 1326
RD	18	708 - 999	549 - 2035	1867 - 2678

Table: Characteristics of the original sets of instances

264 HMRPP instances

Computational results

Set	hier	opt	Gap	Time	Nodes	Conn.	Parity	KC
Alba	2	10/10	–	0.5	1.3	27.1	31.3	9.6
	3	10/10	–	1.9	5.4	82.6	55.5	57.2
	4	10/10	–	5.9	12.3	155.0	72.8	122.1
	5	10/10	–	6.5	12.4	175.1	83.6	86.0
Alda	2	10/10	–	5.9	4.6	71.7	104.7	155.4
	3	10/10	–	32.1	29.4	172.0	185.4	358.6
	4	10/10	–	107.9	270.3	315.0	262.0	332.4
	5	10/10	–	64.0	101.8	371.1	264.7	329.3
Madr	2	10/10	–	6.0	6.7	36.3	66.9	92.1
	3	10/10	–	39.0	38.7	91.4	113.3	178.0
	4	9/10	1.26	21.1	649.5	203.9	156.3	202.5
	5	10/10	–	39.1	61.8	213.0	160.5	167.2

Table: Results on the Albaida, Aldaya and Madrigueras instances

Set	hier	opt	Gap	Time	Nodes	Conn.	Parity	KC
RB	2	18/18	–	164.6	87.1	210.1	198.7	509.3
	3	14/18	0.65	500.1	412.1	549.1	451.6	1186.2
	4	12/18	1.20	427.9	650.2	865.1	599.9	1339.4
	5	8/18	1.81	866.3	1442.3	1452.0	722.5	1366.6
RD	2	14/18	1.14	871.9	335.9	390.4	617.7	1255.4
	3	7/18	1.10	432.9	486.7	845.6	837.1	1925.6
	4	3/18	1.71	1669.3	864.0	1526.2	1024.0	1568.3
	5	3/18	1.91	1797.1	788.4	1857.3	1101.0	1480.8

Table: Results on instance sets RB and RD

Computational results: comparison

The only exact method for solving a hierarchical ARP is the one by Cabral, Gendreau, Ghiani & Laporte (2004) for the Hierarchical Chinese Postman Problem. It consists of transforming the HCPP into an equivalent RPP, which is then solved by using the exact method by Ghiani and Laporte (2000).

Since the HCPP is a particular case of the HMRPP, we have compared our B&C with the Cabral et al. procedure on a set of HCPP instances. Since we did not have access to their HCPP instances, we have generated some new ones, again from the RB and RD sets of MRPP instances. Now, all the links are considered required edges to obtain an undirected graph where all the edges are required.

We have transformed these HCPP instances into RPP ones with the Cabral et al. method and solved them with the exact procedure of Corberán et al. (2007)

Computational results: comparison

The transformation creates a new graph containing a copy of each hierarchy, "completed" with shortest paths between every pair of vertices.

This graph contains a huge number of edges, many of which can be removed. Although this simplification is not mentioned in Cabral et al. (2004), it is reasonable to assume that the authors would apply it in instances of the size we want to solve here.

	RB			RD		
H	Unsimplified	Simplified	%	Unsimplified	Simplified	%
2	295227.5	198329.5	67.2	1174152.7	787150.2	67.1
3	406749.5	249406.8	61.3	1652088.8	1005908.2	60.9
4	457057.7	272266.6	59.7	1842069.0	1081384.6	58.8
5	461229.7	271887.9	59.1	1840200.5	1058054.8	57.6

Table: Variables in the RPP instances obtained from the HCPP ones

Computational results: comparison

Results obtained with the Cabral et al. procedure and our B&C.

Column 'LB' represents the average final lower bound for those instances that were not solved by neither the Cabral et al. procedure nor by our B&C.

		Cabral et al.			B & C		
Set	hier	# opt	LB	Time	# opt	LB	Time
RB	2	18/18	–	239.9	18/18	–	24.3
	3	12/18	61639.2	1369.4	16/18	61705.3	766.7
	4	6/18	75344.8	1108.3	6/18	75350.8	1267.9
	5	3/18	85761.1	1982.9	3/18	85750.4	1457.3
RD	2	6/18	–	2120.4	18/18	–	755.5
	3	0/18	97476.0	–	7/18	97530.3	1690.0
	4	0/18	113091.4	–	0/18	114260.9	–
	5	0/18	124103.2	–	0/18	124112.3	–

Table: Comparison with the Cabral et al. algorithm

Conclusions

- We have studied an interesting ARP with hierarchies, the HMRPP, which arises in different contexts such as snow plowing, street cleaning, and garbage collection.
- We have provided the first polyhedral analysis of the HMRPP and proposed several separation heuristics to find violated inequalities to be added in a branch-and-cut algorithm.
- Our B&C has been able to find 198 optimal solutions out of 264 instances in less than one hour.
- We have compared it with the exact procedure by Cabral et al. (2004) for the HCPP that is a particular case of the HMRPP. The results obtained show a slightly better performance of our method.

THANKS FOR YOUR ATTENTION