Exact algorithm for the Reliability Fixed-Charge Location Problem with Capacity constraints

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IX International Workshop on Locational Analysis and Related Problems 2019. Cádiz. January, 2019

RFLP

- Introduced by Snyder & Daskin in *Reliability models for facility location: the expected failure cost case. Transportation Science, 39 (2.005)*.
- **Fixed-charge facility location problem**.
- Unsplittable demands.
- Facilities can independently fail with homogeneous probability.
- For each customer, a sequence of assignments to opened facilities is defined and, at each scenario, the customer is served from the first facility in the sequence that has not failed.
- An extra dummy non-failing facility with large assignment costs is used to model situations where a customer is lost or outsourced.
- Facilities have unlimited capacity.

CRFLP

- *CRFLP* was introduced by Albareda-Sambola et al. in *Introducing capacities in the location of unreliable facilities. European Journal of Operational Research, 259 (2.017)*
- **It could have facilities with limited capacity**.
- Usually, capacities are considered as strict limits. However, in many situations it is possible to increase the capacity of a facility during emergency situations. In this work different models were explored **allowing to serve a demand slightly over the capacity** of the facilities, but keeping a limit on these excesses.
- **Stability** in the assignment of customers to facilities is compulsory. Then, the orders of the assignments of customers to facilities is predefined and provided by the best solution. The **reassignments are not allowed**.

Sets and parameters

Sets

- I: set of customers.
- J: set of possible locations.
- F: subset of locations of J that could fail.
- NF: subset of locations of J that can not fail.

Parameters

- q: probability of fail for each facility F.
- $h_i \geq 0$: demand of each customer $i \in I$.
- $d_{ij} \geq 0$: cost of sending one unity of product from facility $j \in J$ to customer i.
- $\theta_i \geq 0$: non-service cost of customer $i \in I$.
- $f_j \geq 0$: open cost for location j.

CRFLP Model

Variables

- X_j: binary variables indicating if facility j is open.
- Y_{ijr}: binary variables indicating if j is the $(r+1)$-th backup-facility of customer i.

Objective function

$$min \quad \alpha w_1 + (1-\alpha) w_2$$

where α is a value between $[0,1]$, $R = \{0, ..., |F|\}$ and

$$w_1 = \sum_{j \in J} f_j X_j + \sum_{i \in I} \sum_{j \in J} h_i d_{ij} Y_{ij0}$$

$$w_2 = \sum_{i \in I} h_i [\sum_{j \in NF} \sum_{r \in R} d_{ij} q^r Y_{ijr} + \sum_{j \in F} \sum_{r \in R} d_{ij} q^r (1-q) Y_{ijr}].$$

CRFLP Model

Constraints

$$\text{s.a} \quad \sum_{j\in F} Y_{ijr} + \sum_{j\in NF} \sum_{s=0}^{r} Y_{ijs} = 1 \qquad \forall i \in I, r \in R \qquad (1)$$

$$\sum_{r\in R} Y_{ijr} \leq X_j \qquad \forall i \in I, j \in J \qquad (2)$$

$$X_u = 1 \qquad (3)$$

$$\sum_{i\in I} h_i Y_{ij0} \leq Q_j X_j \qquad \forall j \in J \qquad (4)$$

$$\text{Capacity constraints} \qquad (5)$$

$$X_j \in \{0,1\} \qquad \forall j \in J \qquad (6)$$

$$Y_{ijr} \in \{0,1\} \qquad \forall i \in I, j \in J, r \in R \qquad (7)$$

Expected overload

$$\sum_{j\in O(X)} \mathbb{E}\left[\left(\xi_j \cdot \underbrace{\sum_{i\in I} h_i \left(\sum_{r\in R} Y_{ijr} \cdot \prod_{s<r}\left(\sum_{j'\in O(X)} Y_{ij's}(1-\xi_{j'})\right)\right)}_{\text{demand at } j \text{ according to } \xi} - Q_j\right)^+\right]$$

- $O(X) \subset J$ is the set of locations where facilities have been placed
- $\xi_j \sim \mathrm{Bernoulli}(1-q)$ for $j \in O_F(X) = O(X) \cap F$,
- $\xi_j = 1$ for $j \in O_{NF}(X) = O(X) \cap NF$

Modeling expected overload requires full scenario enumeration being hard to solve it even for small instances.

Capacity constraints

1. **CRFLP-S(θ)** is based on staggered capacities.

$$\sum_{s=0}^{r} \sum_{i \in I} h_i Y_{ijs} \leq \theta^r Q_j \qquad \forall j \in J, r > 1$$

this model does not manage the overload.

2. **CRFLP-B1(β)** bounds an upper bound for expected overload, then their solutions could not be optimal.

$$\sum_{j \in F} \sum_{r>0} q^r (1-q) \lambda_{jr} + \sum_{j \in NF} \sum_{r>0} q^r \lambda_{jr} \leq \beta$$

where λ_{jr}: overload at facility j at level r

3. **CRFLP-LR(β)** bounds a linear estimation of expected overload being an approximated model:

$$2.67827 q \bar{\lambda}_{\bullet 1} + 1.66348 q^2 \bar{\lambda}_{\bullet 2} + 1.92325 q^3 \bar{\lambda}_{\bullet 3} + 4.43350 q^4 \bar{\lambda}_{\bullet 4} \leq \beta$$

where $\bar{\lambda}_{\bullet r}$: average overload at level r.

Proposition 1

Proposition

For any set $O(X)$ of open facilities it is possible to obtain an assignment Y such that the expected overload is bounded by β.

Proof:

Given that in any feasible solution the dummy facility is open, the demand that produces excess over the expected overload cannot be served. □

The key

Case A

	r=0	r=1	r=2
i=0	A	B	u
i=1	B	A	u

E(X,Y) = 7.2
non-service cost = 100.0
overcost = 0.0

Case B

	r=0	r=1	r=2
i=0	A	B	u
i=1	B	u	

E(X,Y) = 3.6
non-service cost = 550.0
overcost = 445.3

Case C

	r=0	r=1	r=2
i=0	A	u	
i=1	B	u	

E(X,Y) = 0.0
non-service cost = 1000.0
overcost = 805.3

If we make a new assignment so that the expected overload is bounded:

1. The non-service expected cost increases. This only depends on the assignment dummy cost and it is the same for solutions of the *'same type'*. Later, we will refer it as **cost of the type of solution**.
2. The overcost increases, too. This depends on the specific solution. In the following, we will refer to this overcost as **overcost of the specific solution**.

Partition of J

Let $\mathcal{P} = \{p_0, ..., p_{K-1}\}$ be a partition in K classes of the set J.

Each class p_k contains facilities with:

- same type of service disruption,
- same capacity level.

Example 1

$J = \{A, B, C, u\}$ such that $Q_A = Q_B = Q_C = 50$, $Q_u = \infty$,
$F = \{A, B\}$, $NF = \{C, u\}$
then $\mathcal{P} = \{p_1, p_2, p_u\}$ with $p_1 = \{A, B\}$, $p_2 = \{C\}$ and $p_u = \{u\}$

Let $n(X)$ be the array containing the number of opened facilities for each class. Two solutions X_1 and X_2 are of the same type if $n(X_1) = n(X_2)$.

$O(X)$	$\bar{O}(X)$	$n(X)$
A, u	B, C	(1, 0, 1)
B, u	A, C	(1, 0, 1)
A, B, C, u	$\emptyset$	(2, 1, 1)

Example of the exact approach

$$J = \{A, B, C, u\} \text{ such that}$$
$$Q_A = Q_B = Q_C = 50,\ Q_u = \infty,$$
$$F = \{A, B\},\ NF = \{C, u\}$$
$$\text{then } \mathcal{P} = \{p_1, p_2, p_u\} \text{ with } p_1 = \{A, B\},\ p_2 = \{C\} \text{ and } p_u = \{u\}$$

	Master Problem				From the solution		
It.	v	$O(X)$	$\bar{O}(X)$	n_i	w_i	z_i	Overall
1	100	A, u	B, C	(1, 0, 1)	400	450	550
2	150	B, u	A, C	(1, 0, 1)	*	350	500
3	400	A, C, u	B	(1, 1, 1)	100	150	550
4	410	B, C, u	A	(1, 1, 1)	*	100	510
5	**500**	**B, u**	**A, C**	**(1, 0, 1)**	*	*	**500**

Master Problem

$$\text{(MASTER)} \min \ \alpha w_1 + (1-\alpha) w_2 + Z$$

$$(1)-(4),(6),(7)$$

$$\sum_{r\in R}\sum_{i\in I} h_i d_{iu} q^r Y_{iur} \geq W \tag{8}$$

$$W \text{ constraints} \tag{9}$$

$$Z \text{ constraints} \tag{10}$$

$$W, Z \in \mathbb{R} \tag{11}$$

- W represents **the cost of the type of solution**
- Z represents **the overcost of the specific solution**.

W constraints

$$\sum_{r\in R}\sum_{i\in I} h_i d_{iu} q^r Y_{iur} \geq W \tag{12}$$

$$\sum_{j\in p_k} X_j = \sum_{i=0}^{|p_k|} i c_i^k; \qquad \sum_{i=0}^{|p_k|} c_i^k = 1;\, p_k \in \mathcal{P} \tag{13}$$

$$\sum_{k=0}^{|\mathcal{P}|-1} c_{n_k}^k - b_a \leq |\mathcal{P}| - 1 \tag{14}$$

$$\sum_{i=1}^{a} b_i \leq 1; \qquad \sum_{i=1}^{a} w_i b_i = W; \tag{15}$$

$$c_i^k \in \{0,1\};\, b_1 \in \{0,1\};\; k \in \{0,\ldots,|\mathcal{P}|-1\}, i \in \{0,\ldots,|p_k|\} \tag{16}$$

w_i **is the cost of the type of solution.**

W constraints

Example 2

Suppose for *Example 1*:

- 2 previous iterations: $w_1 = 10.5$ and $w_2 = 7.5$ were obtained,
- current iteration (#3): $n(X) = (2, 0, 1)$ and $w_3 = 6.5$.

Then, the constraints to be added are:

$$c_2^0 + c_0^1 + c_1^2 - b_3 \leq 2$$
$$b_1 + b_2 + b_3 \leq 1$$
$$10.5b_1 + 7.5b_2 + 6.5b_3 = W$$
$$b_3 \in \{0, 1\}$$

Z constraints

$$(\text{MASTER}) \quad \min \ \alpha w_1 + (1-\alpha) w_2 + Z$$

$$\sum_{j \in J \setminus O(X)} z X_j + Z \geq z$$

Example 3

	r=0	r=1	r=2
i=0	A	B	u
i=1	B	A	u

E(X,Y) = 7.2
non-service cost = 100.0
overcost = 0.0

	r=0	r=1	r=2
i=0	A	B	u
i=1	B	u	

E(X,Y) = 3.6
non-service cost = 550.0
overcost = 445.3

Target: E(X, Y) $<$ 4 (β = 4)

$$J = \{A, B, C, u\}, \ O(X) = \{A, u\}, \ \bar{O}(\mathbf{X}) = \{\mathbf{B}, \mathbf{C}\}$$

$$445.3 X_B + 445.3 X_C + Z \geq 445.3$$

Slave problem AP-D(X)

$$\min \sum_{i\in I} h_i d_{iu} \sum_{r\geq 1} q^r Y'_{iur}$$

$$\text{s.t.} \sum_{j\in O_F(X)} Y'_{ijr} + \sum_{j\in O_{NF}(X)} \sum_{s=0}^{r} Y'_{ijs} = 1 \qquad i\in I, r\in R \tag{17}$$

$$\sum_{r\in R} Y'_{ijr} \leq 1 \qquad i\in I, j\in O(X) \tag{18}$$

$$h_i(\xi_j^s Y'_{ijr} - \sum_{k\in O(X):k\neq j} \sum_{t=0}^{r-1} \xi_k^s Y'_{ikt}) \leq \delta_{ij}^s \quad i\in I, j\in O(X), r\in R, s\in S \tag{19}$$

$$\sum_{i\in I} \delta_{ij}^s - Q_j \leq \theta_j^s \qquad j\in O(X), s\in S \tag{20}$$

$$\sum_{j\in O(X)} \sum_{s\in S} p^s \theta_j^s \leq B \tag{21}$$

$$Y'_{ijr} \in \{0,1\} \qquad i\in I, j\in O(X), r\in R \tag{22}$$

$$\delta_{ij}^s, \theta_j^s \in \mathbb{R} \qquad i\in I, j\in O(X), r\in R \tag{23}$$

$AP-D(X)$ model is based on scenarios taken into account only the open facilities, then the number of combinations has been highly reduced.

Slave problem AP-A(X)

$$\min \ \alpha \left(\sum_{i \in I} \sum_{j \in O(X)} h_i d_{ij} Y'_{ij0} \right) + (1-\alpha) \sum_{i \in I} h_i \left(\sum_{j \in O_{NF}(X)} \sum_{r \in R} d_{ij} q^r Y'_{ijr} + \sum_{j \in O_F(X)} \sum_{r \in R} d_{ij} q^r (1-q) Y'_{ijr} \right)$$

s.t. $(17) - (23)$

Proposition 2

Proposition

If the non-service costs are higher than other assignment costs, i.e., $d_{iu} > d_{ij}$ for all $i \in I$, $j \in J \setminus \{u\}$, then any optimal solution of AP-A(X) is also an optimal solution of AP-D(X).

Implications: in cases which $d_{iu} > d_{ij}$ (usual cases) we also employ $AP - A(X)$ for obtaining the minimum non-service expected cost capturing the assignments to the dummy facility. Then, we do not solve $AP - D(X)$ problem for these cases.

Algorithm 1: $(v^*, \mathbf{X}^*, \mathbf{Y}^*)$ = main_algorithm()

```
master = model(QRFLP);
(c, X, Y) = solve(master);
overload = E(X,Y);
a = 1;
if overload > β then
    v* = +inf;
else
    v* = c;
    (X*, Y*) = (X, Y)
while v* > c & time < t_limit do
    (w_a, z, Y') = new_assignments(O(X) , c, β);
    add_overcost(master, z, X);
    if c + v < v* then
        v* = c + z;
        (X*, Y*) = (X, Y')
    if is_new_distribution_of_open_facilities(O(X)) then
        add_min_non_service_cost(master, O(X) , P, w, a);
        a = a + 1;
    (c, X, Y) = solve(master);
return (v*, X*, Y*)
```

Generated instances

	♯	Instance ♯	Customers	|F|	$\|NF\|^{(*)}$	q	f_F ($\times 1000$)	f_{NF}/f_F
S20_50_a	180	1 - 10	20	50	0	0.05, 0.10, 0.20	1, 2, 3	1, 2
S20_50_b	180	1 - 10	20	35	15	0.05, 0.10, 0.20	1, 2, 3	1, 2
S50_50_a	10	1 - 10	50	50	0	0.05	2	2
S50_50_b	10	1 - 10	50	35	15	0.05	2	2
S20_75_a	10	11 - 20	20	75	0	0.05	2	2
S20_75_b	10	11 - 20	20	45	30	0.05	2	2

$^{(*)}$: excluding dummy

Instances built from the capacitated p-median instances of the OR-LIBRARY.

$\alpha = 0.5$, non-service cost: $\rho = 400$.

Average values S20_50 instances

	v^*	$E(X, Y)$	$\mathbb{P}$(overload)	Dummy	# Open	Time
MASTER	8997.20	5.19	0.07	0.26	3.44	7.75
CRFLP-B1(3)	9378.19	1.64	0.06	1.92	3.67	30.20
CRFLP-B1(6)	9143.23	3.90	0.07	0.94	3.52	32.33
CRFLP-LR(3)	9287.02	2.53	0.07	1.68	3.57	31.71
CRFLP-LR(6)	9051.44	4.65	0.07	0.56	3.47	17.82
CRFLP-EX(3)	9313.96	2.15	0.10	1.52	3.63	62.82
CRFLP-EX(6)	9143.23	3.90	0.07	0.94	3.52	23.13

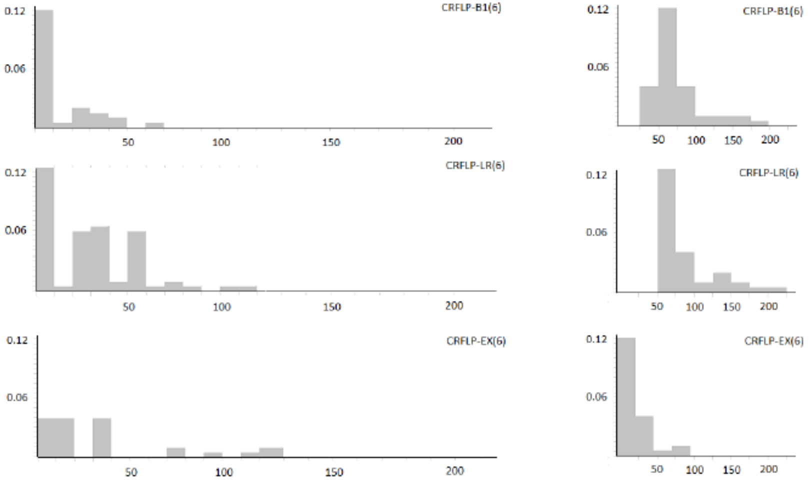
Average values S50_50 instances

	v^*	$E(X, Y)$	$\mathbb{P}$(overload)	Dummy	#Open	Time	Solved
MASTER	17955.67	22.06	0.23	0.00	6.10	719.03	18
CRFLP-B1(3)	21428.69	2.33	0.15	16.37	6.75	187.00	20
CRFLP-B1(6)	20870.85	4.71	0.19	13.37	6.75	215.00	20
CRFLP-LR(3)	21198.76	3.31	0.16	15.12	6.75	1152.60	18
CRFLP-LR(6)	20449.32	6.41	0.21	11.07	6.75	1188.75	16
CRFLP-EX(3)	19961.25	3.00	0.21	10.37	6.10	3600.00	2
CRFLP-EX(6)	19671.84	5.99	0.22	8.68	6.10	3600.00	3

Average values S20_75 instances

	v^*	$E(X, Y)$	$\mathbb{P}$(overload)	Dummy	#Open	Time	Solved
MASTER	9806.13	5.4	0.12	0.22	3.6	349.25	20
CRFLP-B1(3)	11090.57	1.32	0.10	2.44	4.4	1412.03	19
CRFLP-LR(3)	10885.10	2.57	0.12	1.91	4.3	1337.10	17
CRFLP-EX(3)	10496.21	2.84	0.15	1.17	4.1	1398.65	18

Distribution of overload and non-service demand



Conclusions

- We have proposed a **dynamic approach** in order to provide the best solution strictly bounding the expected overload of the *CRFLP*.
- The approach proposed highly **reduces the combinatorial difficulty** inherent to the problem by:
  - Introducing the minimum overcost for each combination among facilities of the same type (W constraints).
  - Associating the overcost due to the assignment for a given set of open facilities that bound the overload (Z constraints).
- This approach has provided the most promising results not only in terms of the cost for strictly limiting the expected overload, but **in many cases even in less time that the approximated methods**, too.
- Even in all of the instances in which the exact method has not finished in time, **the solution returned has worked better than the solution provided by the other methods**.

Future research lines

- We can consider **developing heuristic algorithms** based on this dynamic method, and so applying these for solving larger instances.
- Since the idea of controlling the expected overload by exact dynamic approach has worked efficiently, we can also consider **extending this idea to other reliability problems**.

Some references

- Snyder, L. & Daskin, M.S. (2005). Reliability models for facility location: The expected failure cost case. Transportation Science, 39 (3), 400 416 .
- Albareda-Sambola, M. & Landete, M. & Monge, J.F. & Sainz-Pardo, J.L. (2017) Introducing capacities in the location of unreliable facilities. European Journal of Operational Research, 259, 175 - 188
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Thanks!