New bilevel programming approaches to the location of controversial facilities

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Contents

Contents

"Location is an area of research aimed at determining the optimal locations of suppliers of services, facilities, structures or objects of a certain type, so that certain goals are achieved. The aims pursued depend on the location of the object may be, for example, maximizing the benefits from the operation of the facilities to be located, to minimize travel costs, maximization of the population covered by a service points or minimizing these facilities produced by unwanted or adverse effects."

Redloca, (Location and Related Problems Network in Spain).

Controversy in location problems

Controversy in location problems

Motivated by:

- Different decision-makers with distinct or opposite objectives.
- Conflict of interests between users and decision-makers.
- By the own nature of the of the facility or service.
- Etc.

Controversy in location problems

- Hammad, A. W., Rey, D., and Akbarnezhad, A. (2018). A Bi-level Mixed Integer Programming Model to Solve the Multi-Servicing Facility Location Problem, Minimising Negative Impacts Due to an Existing **Semi-Obnoxious** Facility. Data and Decision Sciences in Action, 381-395. Springer, Cham.
- Heydari, R., and Melachrinoudis, E. (2012). *Location of a* **semi-obnoxious** *facility with elliptic maximin and network minisum objectives*. European journal of operational research, 223(2), 452-460.
- Yapicioglu, H., Smith, A. E., and Dozier, G. (2007). *Solving the* **semi-desirable** *facility location problem using bi-objective particle swarm*. European Journal of Operational Research, 177(2), 733-749.
- Melachrinoudis, E., and Xanthopulos, Z. (2003). **Semi-obnoxious** *single facility location in Euclidean space*. Computers and Operations Research, 30(14), 2191-2209.
- ...

Controversy in location problems

Semi-obnoxious facilities

useful but unwelcome facilities that produce environmental concerns. Facilities that population centres want them away, but there are some interests, or some interested people in locating them close the demand points.

Brimberg, J., and Juel, H. (1998) A bi-criteria model for locating a semi-desirable facility in the plane. European Journal of Operational Research 106 (1), 144–151.

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Controversy in location problems

- Akbari-Jafarabadi, M., Tavakkoli-Moghaddam, R., Mahmoodjanloo, M., and Rahimi, Y. (2017). A tri-level **r-interdiction** median model for a facility location problem under imminent attack. Computers and Industrial Engineering, 114, 151-165.
- Aksen, D., and Aras, N. (2012).A bilevel fixed charge location model for **facilities under imminent attack**. Computers and Operations Research, 39(7), 1364-1381.
- Church, R. L., and Scaparra, M. P. (2007). Protecting critical assets: the **r-interdiction** median problem with fortification. Geographical Analysis, 39(2), 129-146.
- Scaparra, M. P., and Church, R. L. (2008). A bilevel mixed-integer program for **critical infrastructure** protection planning. Computers and Operations Research, 35(6), 1905-1923.
- ...

These problems have been usually addressed via

- biobjective (multiobjective) approaches,
- difference of objectives,
- maximin or minmax objectives,
- bilevel optimization.

Bilevel Programming targets hierarchical optimization problems in which part of the constraints translate the fact that some of the variables constitute an optimal solution to another nested optimization problem.

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$$
\begin{array}{lll}
\max & f(x,y) & \textit{leader objective function} \\
\text{st.} & & \\
& (x,y) \in X & \textit{leader constraints} \\
& & \\
& y \in \mathit{arg}\max\limits_{y} g(x,y) & \textit{follower objective function} \\
& \qquad\qquad \text{st.}(x,y) \in Y & \textit{follower constraints}
\end{array}
$$

First mathematical model by Bracken and McGill (1972).

Model motivation

Model motivation

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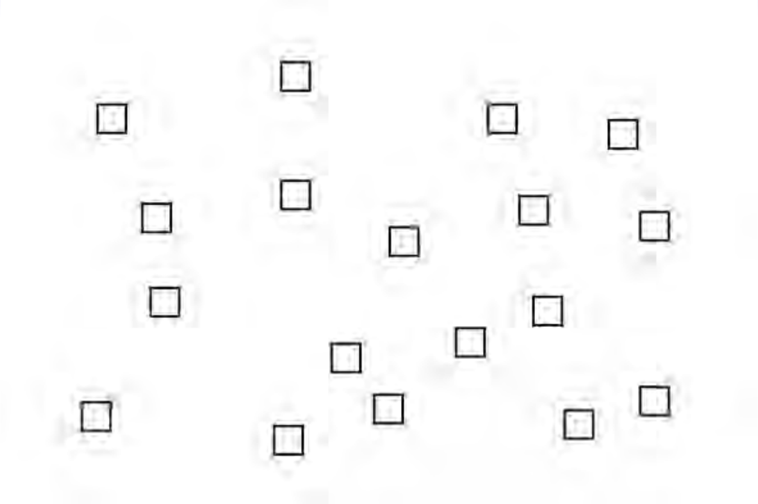
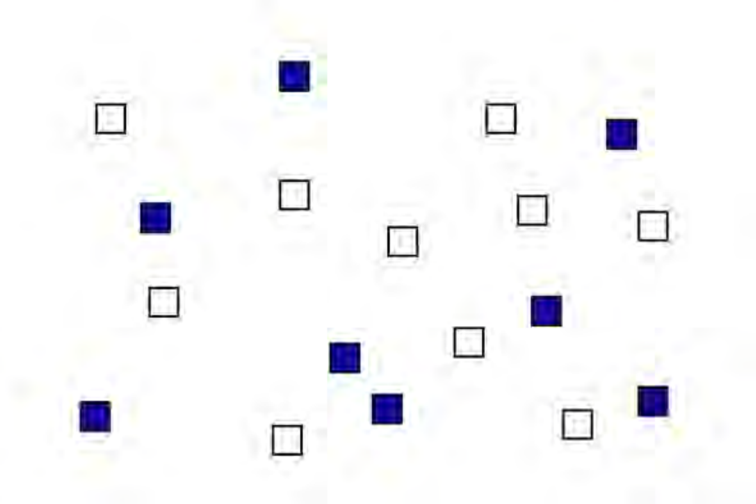


Model motivation



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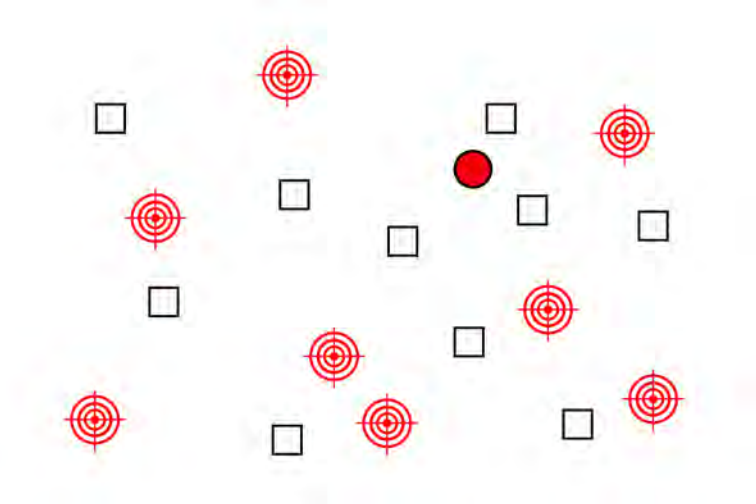

Contents

The model

Location entities

Hierarchy

Aim

The model

Location entities | Hierarchy | Aim

- **Primary facilities** (critical infraestructures, goods to protect, demand points,...)

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- **Primary facilities** (critical infraestructures, goods to protect, demand points,...)

- **Secondary facilities** (attackers, thefts, semi-obnoxious facilities, ...)

The model

Location entities	Hierarchy	Aim
• **Primary facilities** (critical infraestructures, goods to protect, demand points,...)	• **First**, from a discrete set of possible location	
• **Secondary facilities** (attackers, thefts, semi-obnoxious facilities, ...)		

The model

Location entities	Hierarchy	Aim
• **Primary facilities** (critical infraestructures, goods to protect, demand points,...)	• **First**, from a discrete set of possible location	
• **Secondary facilities** (attackers, thefts, semi-obnoxious facilities, ...)	• **Second**, in a continuous framework	

The model

Location entities	Hierarchy	Aim
• **Primary facilities** (critical infraestructures, goods to protect, demand points,...)	• **First**, from a discrete set of possible location	• **Far** from secondary facilities
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The model

Location entities	Hierarchy	Aim
Primary facilities (critical infraestructures, goods to protect, demand points,...)	**First**, from a discrete set of possible location	**Far** from secondary facilities
Secondary facilities (attackers, thefts, semi-obnoxious facilities, ...)	**Second**, in a continuous framework	**Close** to primary facilities

The Model

Upper level:

Lower level:

The Model

Upper level:

- B the set of indexes of potential locations.
- NB the set indexes of already fixed locations.
- $\forall j \in B \cup NB$, $f_j = (f_{j1}, ..., f_{jn})$ the potential location, where n is the dimension of the space.
- Binary decision variables $y_j = 1$ if f_j, $j \in B$ is established, and $y_j = 0$ otherwise.
- c_j the cost of setting the primary point j, for all $j \in B$, and C the maximum budget.

Lower level:

The Model

Upper level:

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Lower level:

- Decision variable $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ that determines the location of the secondary point

The Model

$$\max \sum_{j \in B} w_j d_p(x, f_j) y_j + \sum_{j \in NB} w_j d_p(x, f_j) \qquad \textbf{(BLM)}$$

st.

$$\sum_{j \in B} c_j y_j \leq C,$$

$$y_j \in \{0, 1\} \; \forall j \in B,$$

$$x \in argmin_x \sum_{j \in B} w_j d_p(x, f_j) y_j + \sum_{j \in NB} w_j d_p(x, f_j),$$

where $w_p \in \mathbb{R}$ and $d_p(x, f_j)$ denotes any distance induced by a norm:

$$d_p(x, f_j) = ||x - f_j||_p.$$

The Model

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Theorem

The **BLM** *is NP-hard.*

The Model

We assume that:

- we measure distances via **block norms** (polyhedral norms): norm $||.||_p$ whose unit ball P is a polyhedral set,

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- we measure distances via **block norms** (polyhedral norms): norm $||.||_p$ whose unit ball P is a polyhedral set,
- we are given a polytope P (convex set containing the origin in its interior, symmetry with respect to the origin),
- $\text{ext}(P)$ is the set of extreme points, and
- P^o is the polar set of P,

$$P^o = \{x \in \mathbb{R}^n \ : < x, p > \leq 1, \ \forall p \in P\}.$$

Contents

The model

Norm induced by P

$$||x||_p = \min \sum_{u \in \text{ext}(P)} \mu_u,$$

$$st.\ x = \sum_{u \in \text{ext}(P)} \mu_u u,$$

$$\mu_u \geq 0, \quad u \in \text{ext}(P).$$

The model

Norm induced by P

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$$st.\ x = \sum_{u \in \text{ext}(P)} \mu_u u,$$
$$\mu_u \geq 0, \quad u \in \text{ext}(P).$$

Norm induced by P^o

$$||x||_P = \max_{u \in \text{ext}(P^0)} < x, u > .$$

Rockafellar, R. T. (1970). *Convex analysis*. Princeton university press.

Formulation 1 (Norm induced by P)

Norm induced by P

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\begin{aligned}
||x||_p = \min & \sum_{u \in \mathrm{ext}(P)} \mu_u, \\
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$$\begin{aligned} ||x||_p = \min\ & r \\ st.\ & r = \sum_{u \in \text{ext}(P)} \mu_u, \\ & x = \sum_{u \in \text{ext}(P)} \mu_u u, \\ & \mu_u \geq 0, \quad u \in \text{ext}(P). \end{aligned}$$

Formulation 1 (Norm induced by P)

$$
\begin{aligned}
\max \sum_{j\in B} w_j y_j r_j &+ \sum_{j\in NB} w_j r_j \\
\text{s.t.} \sum_{j\in B} c_j y_j &\leq C, \\
y_j &\in \{0,1\} \quad j \in B, \\
r &\in argmin_{x,r} \sum_{j\in B} w_j y_j r_j + \sum_{j\in NB} w_j r_j, \\
x_i &\in \mathbb{R}, \quad i = 1, ..., n, \\
r_j &= \sum_{u\in ext(P)} \mu_u^j, \quad j \in B \cup NB, \\
x_i &= \sum_{u\in ext(P)} \mu_u^j u_i + f_{ji}, \quad j \in B \cup NB, i = 1, ..., n, \\
\mu_u^j &\geq 0, \quad u \in ext(P), j \in B \cup NB, \\
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Discrete location leader problem (budget constraint)

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Discrete location leader problem (budget constraint)

Continuous location follower problem

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Continuous location follower problem

Representation of the norm

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$$\mu_u^j \ge 0, \quad u \in ext(P), j \in B \cup NB,$$

$$r_j \ge 0, \quad j \in B \cup NB.$$

Dual:

$$\max \sum_{i=1}^{n} \sum_{j\in B\cup NB} \beta_{ji} f_{ji}$$

$$\text{st.} \; \alpha_j \le w_j y_j, \quad j \in B,$$

$$\alpha_j \le w_j, \quad j \in NB,$$

$$\sum_{j\in B\cup NB} \beta_{ji} = 0, \quad i = 1, \dots, n,$$

$$-\alpha_j - \sum_{i=1}^{n} u_i \beta_{ji} \le 0, \; j \in B \cup NB,$$

$$u \in ext(P).$$

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Using strong duality

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Using strong duality

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$$x_i \in \mathbb{R}, \quad i = 1, ..., n,$$

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Linearising $y_j r_j$: MILP Formulation.

$$\sum_{j\in B} w_j y_j r_j + \sum_{j \in NB} w_j r_j = \sum_{i=1}^{n} \sum_{j \in B\cup NB} \beta_{ji} f_{ji}$$

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Formulation 2 (Norm induced by P^0)

Norm induced by P^o

$$||x||_p = \max_{u \in \text{ext}(P^0)} < x, u > .$$

Formulation 2 (Norm induced by P^0)

Norm induced by P°

$$||x||_p = \max_{u \in \text{ext}(P^0)} < x, u > .$$

$$\begin{aligned} ||x||_p = \min \ & r \\ & st. \ < u, x > \leq r, \ \forall u \in ext(P^0). \end{aligned}$$

Formulation 2 (Norm induced by P^0)

$$\max \sum_{j\in B} w_j y_j r_j + \sum_{j\in NB} w_j r_j$$

$$\text{s.t.} \sum_{j\in B} c_j y_j \leq C,$$

$$y_j \in \{0,1\} \quad j \in B,$$

$$\sum_{j\in B} w_j y_j r_j + \sum_{j\in NB} w_j r_j = \sum_{k\in ext(P_0)} \sum_{j\in B\cup NB} (\sum_{i=1}^{n} -f_{ji} u_{ki}) \alpha_{kj},$$

$$r_j \geq \sum_{i=1}^{n} u_{ki}(x_i - f_{ji}), \quad k \in ext(P^0), j \in B \cup NB,$$

$$r_j \geq 0, \quad j \in B \cup NB,$$

$$x_i \in \mathbb{R}, \quad i = 1, \dots, n,$$

$$\sum_{k\in ext(P^0)} \alpha_{kj} \leq w_j y_j, \quad j \in B,$$

$$\sum_{k\in ext(P^0)} \alpha_{kj} \leq w_j, \quad j \in NB,$$

$$\sum_{k\in ext(P^0)} \sum_{j\in B\cup NB} (-u_{ki}) \alpha_{kj} = 0, \quad 1 = 1, \dots n,$$

$$\alpha_{kj} \geq 0 \quad k \in ext(P_0), j \in B \cup NB.$$

Benders like algorithm

Benders like algorithm

$$
\begin{aligned}
\max \sum_{j\in B} w_j y_j r_j &+ \sum_{j\in NB} w_j r_j \\
\text{s.t.} \sum_{j\in B} c_j y_j &\le C, \\
y_j &\in \{0,1\} \quad j \in B, \\
r &\in argmin_{x,r} \sum_{j\in B} w_j y_j r_j + \sum_{j\in NB} w_j r_j, \\
x_i &\in \mathbb{R}, \quad i = 1, \dots, n, \\
r_j &= \sum_{u\in ext(P)} \mu_u^j, \quad j \in B \cup NB, \\
x_i &= \sum_{u\in ext(P)} \mu_u^j u_i + f_{ji}, \quad j \in B \cup NB, i = 1, \dots, n, \\
\mu_u^j &\ge 0, \quad u \in ext(P), j \in B \cup NB, \\
r_j &\ge 0, \quad j \in B \cup NB.
\end{aligned}
$$

Discrete location leader problem (budget constraint)

Continuous location follower problem

Representation of the norm

Benders like algorithm

We denote by $\mathbb{P}$ the set of extreme points of the follower problem (for a norm l_p).

Benders like algorithm

We denote by $\mathbb{P}$ the set of extreme points of the follower problem (for a norm l_p).

Solving such follower problem is equivalent to:

$$\max\ q$$

$$\text{s.t.}$$

$$q \leq \sum_{j \in B} \sum_{i=1}^{n} w_j y_j r_j^\tau + \sum_{j \in NB} \sum_{i=1}^{n} w_j r_j^\tau, \qquad r^\tau \in \mathbb{P}.$$

Benders like algorithm

Model rewriting:

$$\max \ q$$

$$\sum_{j \in B} c_j y_j \leq C,$$

$$y_j \in \{0,1\}, \quad j \in B,$$

$$q \leq \sum_{j \in B} \sum_{i=1}^{n} w_j y_j r_j^{\tau} + \sum_{j \in NB} \sum_{i=1}^{n} w_j r_j^{\tau}, \ \ r^{\tau} \in \mathbb{P},$$

Benders like algorithm

Master Problem:

$$\max \ q$$

$$q \leq \sum_{j \in B} \sum_{i=1}^{n} w_j y_j r_j^{\tau} + \sum_{j \in NB} \sum_{i=1}^{n} w_j r_j^{\tau}, \ \tau \in \mathcal{P},$$

$$\sum_{j \in B} c_j y_j \leq C,$$

$$y_j \in \{0,1\} \ \forall j \in B,$$

Benders like algorithm

Primal Problem:

$$
\begin{aligned}
\min\ q(y) = \sum_{j \in B} w_j y_j r_j + \sum_{j \in NB} w_j r_j,\\
r_j = \sum_{e \in ext(P)} \mu_e^j, \quad j \in B \cup NB,\\
x_i = \sum_{e \in ext(P)} \mu_e^j e_i + f_{ji}, \quad j \in B \cup NB, i = 1, \ldots, n,\\
\mu_e^j \geq 0, \quad e \in ext(P), j \in B \cup NB,\\
r_j \geq 0, \quad j \in B \cup NB,\\
x_i \in \mathbb{R}, \quad i = 1, \ldots, n.
\end{aligned}
$$

Benders like algorithm

ALGORITHM:

Initialization Choose a solution y^0, and solve the Primal Problem for the chosen y^0. Let z^0 be an optimal solution for the Primal Problem. Take $\mathcal{P} = \{0\}$ and go to iteration $\nu = 1$.

Iteration $\nu = 1, 2, \ldots$ Solve the Master Problem. Let y^* be an optimal solution of such problem and q^+ the corresponding optimal value.

- If $q^+ = q(y^*)$. END.
- Otherwise,solve the Primal Problem for $y = y^*$. Let z^* be an optimal solution of such problem. Take $z^\nu = z^*$, $\mathcal{P} := \mathcal{P} \cup \{\nu\}$, and go to iteration $\nu := \nu + 1$.

Contents

Examples: l_1 and l_∞ norm

l_1 norm (rectangular norm)

$$||x||_1 = \sum_{i=1}^{n} |x_i|.$$

- $B_{l_1} = \{x \in \mathbb{R}^n : \ ||x||_1 \leq 1\}$
- $ext(P) = \{e_1, ..., e_n, -e_1, ..., -e_n\}$

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- $ext(P) = \{e_1, ..., e_n, -e_1, ..., -e_n\}$

- $B_{l_\infty} = \{x \in \mathbb{R}^n : \ ||x||_\infty \le 1\}$
- $ext(P) = \big\{(1^{a_1}, ..., 1^{a_n}) \in \mathbb{R}^n : a_i \in \{1, -1\}, i = 1, ..n\big\}$

Examples: l_1 and l_∞ norm

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- $ext(P^0) = \left\{(1^{a_1}, ..., 1^{a_n}) \in \mathbb{R}^n : a_i \in \{1, -1\}, i = 1, ..n\right\}$

- $B_{l_\infty} = \{x \in \mathbb{R}^n : \ ||x||_\infty \le 1\}$
- $ext(P) = \left\{(1^{a_1}, ..., 1^{a_n}) \in \mathbb{R}^n : a_i \in \{1, -1\}, i = 1, ..n\right\}$
- $ext(P^0) = \{e_1, ..., e_n, -e_1, ..., -e_n\}$

Examples: l_1 and l_∞ norm

$$\max \sum_{j\in B}\sum_{i=1}^{n} w_j \hat{r}_{ij} + \sum_{j\in NB}\sum_{i=1}^{n} w_j r_{ji} \qquad (\mathrm{l}_1 \text{ norm Formulation})$$

$$\text{s.t.} \sum_{j\in B} c_j y_j \le C,$$

$$y_j \in \{0,1\}\ \forall j \in B,$$

$$\sum_{j\in B}\sum_{i=1}^{n} w_j \hat{r}_{ij} + \sum_{j\in NB}\sum_{i=1}^{n} w_j r_{ji} = \sum_{j\in B\cup NB}\sum_{i=1}^{n} -f_{ji}u_{ji} + \sum_{j\in B}\sum_{i=1}^{n} f_{ji}(w_j y_j - u_{ji}) + \sum_{j\in NB}\sum_{i=1}^{n} f_{ji}(w_j - u_{ji})$$

$$r_{ji} \ge x_i - f_{ji}, \quad j \in B\cup NB, i = 1,\dots,n,$$

$$r_{ji} \ge f_{ji} - x_i, \quad j \in B\cup NB, i = 1,\dots,n,$$

$$\hat{r}_{ji} \le My_j,, \quad j \in B, i = 1,\dots,n,$$

$$\hat{r}_{ji} \le r_{ji},, \quad j \in B, i = 1,\dots,n,$$

$$\hat{r}_{ji} \ge r_{ji} - (1-y_j)M,, \quad j \in B, i = 1,\dots,n,$$

$$\hat{r}_{ji} \ge 0,, \quad j \in B, i = 1,\dots,n,$$

$$\sum_{j\in B}\left(-2u_{ji} + w_j y_j\right) + \sum_{j\in NB}\left(-2u_{ji} + w_j\right) = 0, \quad i = 1,\dots,n,$$

$$u_{ji} \ge 0, \quad j \in B\cup NB, i = 1,\dots,n,$$

$$w_j y_j - u_{ji} \ge 0, \quad j \in B, i = 1,\dots,n,$$

$$w_j - u_{ji} \ge 0, \quad j \in B, i = 1,\dots,n.$$

Examples: l_1 and l_∞ norm

The inner location problem can be decomposed into n independent linear programs, one per each coordinate.

Examples: l_1 and l_∞ norm

The inner location problem can be decomposed into n independent linear programs, one per each coordinate.

MILP formulation:

$$\sum_{j\in B} w_j \hat{r}_{ji} + \sum_{j\in NB} w_j r_{ji} = \sum_{j\in B\cup NB} -f_{ji}\alpha_{ji} + \sum_{j\in B} f_{ji}(w_j y_j - \alpha_{ji}) + \sum_{j\in NB} f_{ji}(w_j - \alpha_{ji}), \quad i = 1, ..., n.$$

Examples: l_1 and l_∞ norm

The inner location problem can be decomposed into n independent linear programs, one per each coordinate.

MILP formulation:

$$\sum_{j\in B} w_j \hat{r}_{ji} + \sum_{j\in NB} w_j r_{ji} = \sum_{j\in B\cup NB} -f_{ji}\alpha_{ji} + \sum_{j\in B} f_{ji}(w_j y_j - \alpha_{ji}) + \sum_{j\in NB} f_{ji}(w_j - \alpha_{ji}), \quad i = 1, \dots, n.$$

Benders like algorithm:

$$\begin{aligned} \max \quad & \sum_{i=1}^n q_i \\ \text{s.t.} \quad & \\ & q_i \le \sum_{j\in B} w_j y_j z_{ij}^{\tau} + \sum_{j\in NB} w_j z_{ij}^{\tau} \quad \forall \tau \in \mathcal{P}, \quad \forall i = 1, \dots, n, \\ & \sum_{j\in B} c_j y_j \le C, \\ & y_j \in \{0,1\} \ \forall j \in B, \end{aligned}$$

Examples: l_1 and l_∞ norm

$$
\begin{aligned}
\max \; & \sum_{j\in B} w_j \hat{r}_j + \sum_{j\in NB} w_j r_j && (\text{l}_\infty \text{ norm Formulation})\\
\text{s.t.} \; & \sum_{j\in B} c_j y_j \leq C, \\
& y_j \in \{0,1\} \; \forall j \in B, \\
& \sum_{j\in B} w_j \hat{r}_j + \sum_{j\in NB} w_j r_j = \sum_{j\in B\cup NB} \sum_{i=1}^{n} f_{ji}(-u_{ji} + v_{ji}), \\
& r_j \geq x_i - f_{ji}, \quad j \in B \cup NB, i = 1, \dots, n, \\
& r_j \geq f_{ji} - x_i, \quad j \in B \cup NB, i = 1, \dots, n, \\
& \sum_{j\in B\cup NB} (-u_{ji} + v_{ji}) = 0, \quad i = 1, \dots, n, \\
& \sum_{i=1}^{n} (u_{ji} + v_{ji}) = w_j y_j, \quad j \in B, \\
& \sum_{i=1}^{n} (u_{ji} + v_{ji}) = w_j, \quad j \in NB, \\
& u_{ji} \geq 0, v_{ji} \geq 0, \quad j \in B \cup NB, i = 1, \dots, n, \\
& \hat{r}_j \leq M y_j, \quad j \in B \\
& \hat{r}_j \leq r_j, \quad j \in B, \\
& \hat{r}_j \geq r_j - (1 - y_j)M, \hat{r}_j \geq 0 \quad j \in B.
\end{aligned}
$$

Contents

Computational Results

- **n** (dimension): **2,3,10,20**.
- **card(B)** (potential facilities): **1000, 2000, 5000, 10000**.
- **card(NB)** (opened facilities): $\{\mathbf{\frac{1}{4}}, \mathbf{\frac{1}{3}}, \mathbf{\frac{1}{2}}\}$**card(B)**.
- $\mathbf{w_j}$ (proportional charge): **random in** $[\mathbf{0}, \mathbf{1}]$.
- $\mathbf{c_j}$ (opening cost): **random in** $[\mathbf{0}, \mathbf{1}]$.
- **C** (budget):$\{\mathbf{\frac{1}{4}}, \mathbf{\frac{1}{3}}\}$**card(B)**.
- $\mathbf{f_{ij}}$ (coordinates): **random in** $[-\mathbf{1000}, \mathbf{1000}]$.

Computational Results

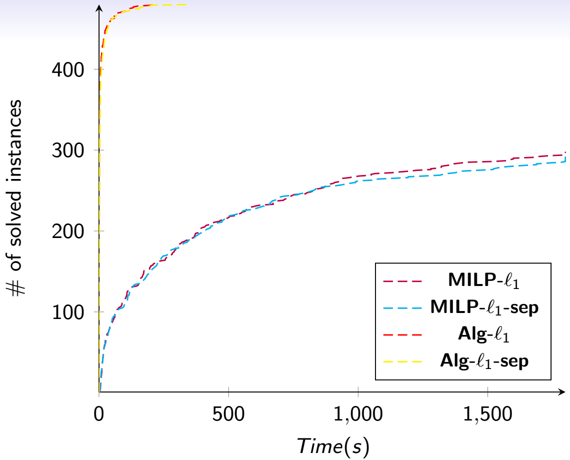

Figure: Performance profile graph of #solved instances for the different proposed models for the ℓ_1 norm.

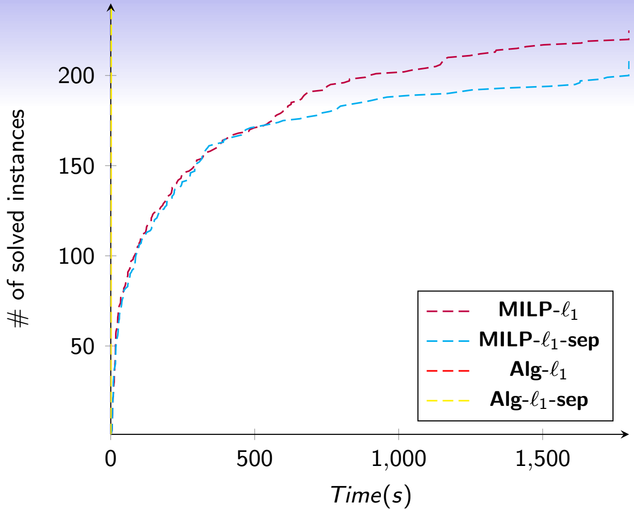

Figure: Performance profile graph of #solved instances for the different proposed models for *small* instances ($n = 2, 3$) for the ℓ_1 norm.

Computational Results

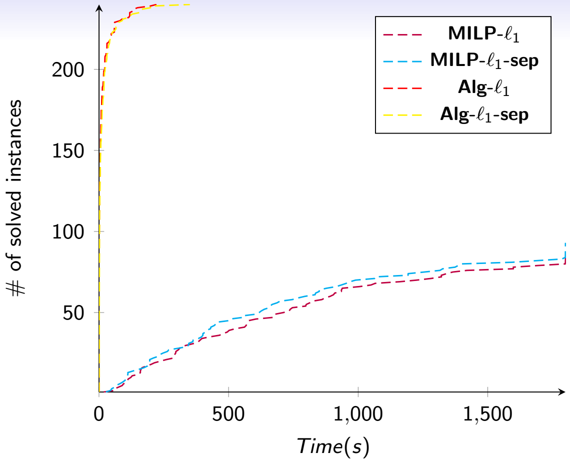

Figure: Performance profile graph of #solved instances for the different proposed models for *big* instances ($n = 10, 20$) for the ℓ_1 norm.

Computational Results

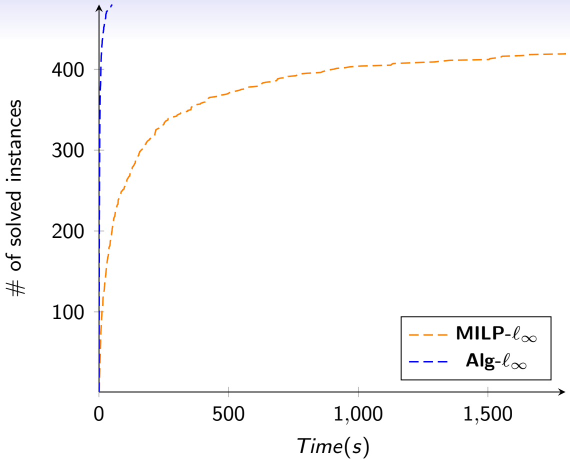

Figure: Performance profile graph of #solved instances for the different proposed models for the ℓ_∞ norm.

INSTANCES				MILP-ℓ_1		MILP-ℓ_1-sep		Alg-ℓ_1		Alg-ℓ_1-sep		MILP-ℓ_∞		Alg-ℓ_∞	
n	|B|	|NB|	C'	#OPT	CPU	#OPT	CPU	#OPT	CPU	#OPT	CPU	#OPT	CPU	#OPT	CPU
2	1000	250	3	5	10,82	5	10,36	5	0,41	5	0,50	5	2,13	5	0,56
2	1000	250	4	5	4,80	5	5,89	5	0,62	5	0,52	5	2,22	5	1,03
2	1000	333	3	5	13,39	5	10,07	5	0,35	5	0,41	5	2,19	5	0,65
2	1000	333	4	5	7,57	5	4,98	5	0,55	5	0,62	5	2,47	5	1,05
2	1000	500	3	5	11,70	5	12,76	5	0,38	5	0,43	5	8,53	5	0,81
2	1000	500	4	5	7,86	5	6,35	5	0,54	5	0,52	5	3,02	5	0,95
2	2000	500	3	5	36,59	5	37,27	5	0,75	5	1,44	5	6,34	5	0,79
2	2000	500	4	5	13,36	5	11,66	5	0,65	5	0,73	5	9,15	5	1,36
2	2000	667	3	5	217,01	5	152,79	5	0,68	5	0,85	5	7,20	5	0,92
2	2000	667	4	5	20,58	5	14,54	5	0,45	5	0,60	5	6,14	5	1,17
2	2000	1000	3	5	56,48	5	78,63	5	0,45	5	0,61	5	15,29	5	0,81
2	2000	1000	4	5	29,85	5	20,60	5	0,52	5	0,65	5	12,74	5	1,56
2	5000	1250	3	5	204,62	5	508,34	5	1,19	5	1,58	5	48,51	5	1,62
2	5000	1250	4	5	124,68	5	97,10	5	1,49	5	2,36	5	52,24	5	1,70
2	5000	1667	3	5	381,53	5	268,18	5	1,03	5	1,32	5	45,06	5	1,47
2	5000	1667	4	5	135,94	5	198,89	5	2,08	5	1,66	5	37,01	5	1,96
2	5000	2500	3	5	416,01	5	369,43	5	0,80	5	1,12	5	139,15	5	2,34
2	5000	2500	4	5	107,17	5	405,78	5	1,23	5	1,89	5	38,32	5	1,92
2	10000	2500	3	4	708,80	3	955,21	5	1,68	5	2,57	5	137,97	5	3,29
2	10000	2500	4	5	325,03	5	355,51	5	2,44	5	3,75	5	81,66	5	3,72
2	10000	3333	3	5	390,82	5	296,18	5	1,94	5	2,78	5	121,14	5	3,42
2	10000	3333	4	5	329,09	5	439,71	5	3,09	5	3,46	5	159,98	5	3,16
2	10000	5000	3	5	506,03	4	671,22	5	2,16	5	2,09	4	530,49	5	3,38
2	10000	5000	4	5	516,73	5	416,86	5	2,83	5	3,27	5	141,33	5	3,77

Table: Numerical results for $n = 2$ under the l_1 and ell_∞ norm.

INSTANCES				MILP-ℓ_1		MILP-ℓ_1-sep		Alg-ℓ_1		Alg-ℓ_1-sep		MILP-ℓ_∞		Alg-ℓ_∞	
n	|B|	|NB|	C'	#OPT	CPU	#OPT	CPU	#OPT	CPU	#OPT	CPU	#OPT	CPU	#OPT	CPU
3	1000	250	3	5	9,67	5	19,03	5	0,21	5	0,27	5	5,95	5	0,70
3	1000	250	4	5	58,74	5	43,20	5	0,63	5	0,71	5	3,09	5	1,18
3	1000	333	3	5	21,51	5	50,24	5	0,36	5	0,37	5	6,99	5	0,60
3	1000	333	4	5	29,05	5	32,52	5	0,65	5	0,90	5	4,68	5	1,21
3	1000	500	3	5	26,14	5	28,70	5	0,34	5	0,49	5	10,17	5	0,57
3	1000	500	4	5	33,46	5	47,28	5	0,38	5	0,52	5	6,86	5	0,73
3	2000	500	3	5	120,41	5	144,41	5	0,38	5	0,53	5	63,77	5	1,19
3	2000	500	4	5	389,05	5	301,49	5	0,76	5	1,22	5	21,33	5	1,41
3	2000	667	3	5	316,33	5	232,75	5	1,04	5	0,92	5	31,18	5	1,04
3	2000	667	4	5	129,52	5	224,55	5	0,47	5	0,56	5	21,81	5	1,48
3	2000	1000	3	5	113,90	5	261,64	5	0,65	5	0,53	5	81,42	5	1,38
3	2000	1000	4	5	179,31	5	249,42	5	0,62	5	0,71	5	11,05	5	0,88
3	5000	1250	3	4	1014,05	3	1310,24	5	1,95	5	1,85	5	131,33	5	1,94
3	5000	1250	4	3	1352,73	2	1442,08	5	2,43	5	2,24	5	106,38	5	2,12
3	5000	1667	3	4	763,66	4	669,57	5	1,14	5	1,28	5	180,78	5	1,94
3	5000	1667	4	3	1102,02	4	1047,36	5	1,56	5	1,91	5	213,59	5	2,08
3	5000	2500	3	5	636,19	5	689,33	5	0,76	5	1,14	5	391,97	5	2,07
3	5000	2500	4	5	592,53	2	1223,05	5	1,42	5	1,58	5	77,89	5	1,90
3	10000	2500	3	1	1778,05	1	1800,00	5	2,42	5	3,38	5	899,86	5	3,09
3	10000	2500	4	1	1727,79	0	–	5	5,59	5	5,16	5	198,48	5	4,18
3	10000	3333	3	3	1127,45	1	1732,91	5	2,67	5	3,91	5	832,12	5	3,17
3	10000	3333	4	3	1253,69	2	1321,56	5	4,79	5	6,15	4	797,61	5	3,91
3	10000	5000	3	3	1379,12	1	1771,16	5	2,86	5	4,41	4	470,05	5	4,01
3	10000	5000	4	0	–	0	–	5	4,18	5	5,79	5	325,39	5	4,87

Table: Numerical results for $n = 3$ under the l_1 and ell_∞ norm.

INSTANCES				MILP-ℓ_1		MILP-ℓ_1-sep		Alg-ℓ_1		Alg-ℓ_1-sep		MILP-ℓ_∞		Alg-ℓ_∞	
n	|B|	|NB|	C	#OPT	CPU	#OPT	CPU	#OPT	CPU	#OPT	CPU	#OPT	CPU	#OPT	CPU
10	1000	250	3	5	175,69	5	262,45	5	0,52	5	0,84	5	23,53	5	0,73
10	1000	250	4	5	286,71	5	199,67	5	0,58	5	1,02	5	10,04	5	0,60
10	1000	333	3	5	170,27	5	168,75	5	0,29	5	0,57	5	50,58	5	0,68
10	1000	333	4	5	380,23	5	308,91	5	0,79	5	0,98	5	16,31	5	0,66
10	1000	500	3	5	365,20	5	412,48	5	0,56	5	0,64	5	40,66	5	0,80
10	1000	500	4	4	838,00	5	608,46	5	0,74	5	1,17	5	26,14	5	1,16
10	2000	500	3	3	1538,42	3	1310,11	5	0,81	5	1,90	5	104,57	5	1,31
10	2000	500	4	3	1197,88	4	1020,28	5	1,04	5	1,44	5	64,34	5	1,62
10	2000	667	3	2	1406,81	2	1365,28	5	1,23	5	1,20	5	123,13	5	1,22
10	2000	667	4	3	1092,84	4	928,63	5	1,01	5	1,24	5	58,13	5	1,16
10	2000	1000	3	4	1155,24	4	1125,38	5	0,72	5	0,83	5	183,55	5	1,59
10	2000	1000	4	3	980,54	3	1215,99	5	1,11	5	1,84	5	91,77	5	1,66
10	5000	1250	3	1	1673,56	1	1796,25	5	2,66	5	3,00	3	1265,41	5	4,80
10	5000	1250	4	0	–	0	–	5	9,46	5	14,29	5	421,92	5	5,76
10	5000	1667	3	0	–	0	–	5	5,08	5	10,21	4	840,57	5	3,86
10	5000	1667	4	2	1430,78	2	1690,69	5	2,77	5	3,19	4	627,48	5	5,12
10	5000	2500	3	1	1568,60	0	–	5	3,91	5	5,34	4	754,36	5	5,44
10	5000	2500	4	1	1727,35	1	1649,57	5	6,04	5	8,31	3	1125,20	5	5,97
10	10000	2500	3	0	–	0	–	5	9,37	5	17,15	2	1211,56	5	10,24
10	10000	2500	4	0	–	0	–	5	16,19	5	26,46	4	955,69	5	8,48
10	10000	3333	3	0	–	0	–	5	17,88	5	20,49	1	1649,16	5	13,44
10	10000	3333	4	0	–	0	–	5	19,62	5	18,73	3	1152,58	5	10,75
10	10000	5000	3	1	1800,00	0	–	5	11,24	5	23,00	2	1800,00	5	10,91
10	10000	5000	4	0	–	0	–	5	28,00	5	16,26	2	1314,08	5	14,30

Table: Numerical results for $n = 10$ under the ℓ_1 and l_∞ norm.

INSTANCES				MILP-ℓ_1		MILP-ℓ_1-sep		Alg-ℓ_1		Alg-ℓ_1-sep		MILP-ℓ_∞		Alg-ℓ_∞	
n	|B|	|NB|	C	#OPT	CPU	#OPT	CPU	#OPT	CPU	#OPT	CPU	#OPT	CPU	#OPT	CPU
20	1000	250	3	4	797,87	5	477,74	5	0,70	5	1,38	5	41,66	5	1,25
20	1000	250	4	5	852,01	5	548,28	5	0,64	5	1,31	5	20,22	5	0,88
20	1000	333	3	4	608,99	5	296,04	5	0,54	5	1,03	5	29,01	5	0,99
20	1000	333	4	4	1026,53	4	773,41	5	1,01	5	1,93	5	39,01	5	1,55
20	1000	500	3	4	881,29	5	412,55	5	0,51	5	1,13	5	55,68	5	1,42
20	1000	500	4	2	1499,86	3	1478,83	5	0,93	5	1,55	5	36,02	5	1,48
20	2000	500	3	1	1629,43	2	1800,00	5	1,93	5	2,21	5	212,07	5	2,08
20	2000	500	4	1	1623,38	2	1634,03	5	3,91	5	2,82	5	217,86	5	2,42
20	2000	667	3	2	1612,20	3	1801,02	5	1,78	5	2,05	5	223,80	5	2,40
20	2000	667	4	3	1421,62	2	1416,62	5	1,34	5	2,44	5	134,67	5	2,64
20	2000	1000	3	0	–	2	1800,00	5	2,45	5	1,86	5	181,95	5	3,02
20	2000	1000	4	1	1800,00	1	1800,00	5	5,09	5	3,08	5	345,70	5	3,44
20	5000	1250	3	0	–	0	–	5	7,20	5	9,37	3	1188,15	5	8,67
20	5000	1250	4	0	–	0	–	5	14,87	5	17,02	5	868,00	5	9,05
20	5000	1667	3	0	–	0	–	5	7,92	5	30,75	2	1513,57	5	8,88
20	5000	1667	4	0	–	0	–	5	7,28	5	18,85	5	1034,49	5	9,46
20	5000	2500	3	0	–	0	–	5	16,23	5	13,53	2	2447,37	5	15,67
20	5000	2500	4	0	–	0	–	5	20,15	5	10,45	2	1585,53	5	13,13
20	10000	2500	3	0	–	0	–	5	52,42	5	65,91	2	1745,86	5	23,08
20	10000	2500	4	0	–	0	–	5	96,02	5	63,30	1	1800,00	5	27,02
20	10000	3333	3	0	–	0	–	5	28,93	5	25,17	2	1750,37	5	22,32
20	10000	3333	4	0	–	0	–	5	125,00	5	93,66	0	–	5	29,79
20	10000	5000	3	0	–	0	–	5	40,39	5	85,61	2	1800,00	5	38,66
20	10000	5000	4	0	–	0	–	5	88,29	5	167,54	1	1800	5	27,76

Table: Numerical results for $n = 20$ under the ℓ_1 and l_∞ norm.

Contents

K-followers

$$
\begin{aligned}
\max \sum_{k \in K} & \left(\sum_{j \in B} w_j^k d(x^k, f_j) y_j + \sum_{j \in NB} w_j^k d(x^k, f_j) \right) \\
\text{s.t. } & \sum_{j \in B} c_j y_j \leq C, \\
& y_j \in \{0, 1\}, \quad j \in B, \\
& x^k \in \arg\min_{x^k} \sum_{j \in B} w_j^k d(x^k, f_j) y_j + \sum_{j \in NB} w_j^k d(x^k, f_j) \quad \forall k = 1, ..., K.
\end{aligned}
$$

K-followers

$$
\begin{aligned}
\max \quad & \sum_{k \in K} q^k \\
\text{s.t.} \quad & q^k \leq \sum_{j \in B} \sum_{i=1}^{n} w_j^k y_j r_j^{k\tau} + \sum_{j \in NB} \sum_{i=1}^{n} w_j^k r_j^{k\tau} \quad \forall \tau \in \mathcal{P}, \forall k \in K, \\
& \sum_{j \in B} c_j y_j \leq C, \\
& y_j \in \{0,1\} \ \forall j \in B.
\end{aligned}
$$

ℓ_τ-norms with $\tau \in \mathbb{Q}$, $\tau \geq 1$

$$
\begin{aligned}
\max \; & \sum_{j \in B} w_j \|x - f_j\|_\tau y_j + \sum_{j \in NB} w_j \|x - f_j\|_\tau \\
\text{st.} \; & \sum_{j \in B} c_j y_j \leq C, \\
& y_j \in \{0, 1\} \quad j \in B, \\
& x \in \arg\min_x \sum_{j \in B} w_j \|x - f_j\|_\tau y_j + \sum_{j \in NB} w_j \|x - f_j\|_\tau .
\end{aligned}
$$

ℓ_τ-norms with $\tau \in \mathbb{Q}$, $\tau \geq 1$

$$
\begin{aligned}
\max & \sum_{j\in B} w_j ||x - f_j||_\tau y_j + \sum_{j\in NB} w_j ||x - f_j||_\tau \\
\text{st.} & \sum_{j\in B} c_j y_j \leq C, \\
& y_j \in \{0,1\} \quad j \in B, \\
& x \in \arg\min_x \sum_{j\in B} w_j ||x - f_j||_\tau y_j + \sum_{j\in NB} w_j ||x - f_j||_\tau .
\end{aligned}
$$

Blanco, V., Puerto, J., Ali, SEHB. (2014). Revisiting several problems and algorithms in continuous location with l_τ norms. Computational Optimization and Applications. 58(3), 563-595.

ℓ_τ-norms with $\tau \in \mathbb{Q}$, $\tau \geq 1$

$$
\begin{aligned}
\max & \sum_{j \in B} w_j ||x - f_j||_\tau y_j + \sum_{j \in NB} w_j ||x - f_j||_\tau \\
\text{s.t.} & \sum_{j \in B} c_j y_j \leq C, \\
& y_j \in \{0, 1\} \quad j \in B, \\
& \sum_{j \in B} w_j ||x - f_j||_\tau y_j + \sum_{j \in NB} w_j ||x - f_j||_\tau = \sum_{j \in B \cup NB} \sum_{i=1}^{n} V_{ji} f_{ji}, \\
& x^+ - x^- - Z_j = f_j, \quad \forall\, j \in B \cup NB, \\
& ||Z_j||_\tau \leq r_j, \quad \forall\, j \in B \cup NB, \\
& x^+, x^- \in \mathbb{R}^n_+,\ Z_j \in \mathbb{R}^n,\ r \in \mathbb{R}^n, \\
& \sum_{j \in B \cup NB} V_{ji} + \lambda^1_i = 0, \quad \forall\, i = 1, \ldots, n, \\
& -\sum_{j \in B \cup NB} V_{ji} + \lambda^2_i = 0, \quad \forall\, i = 1, \ldots, n, \\
& -V_{ji} + \mu_{ji} = 0, \quad \forall\, j \in B \cup NB,\ i = 1, \ldots, n, \\
& ||\mu_j||_\rho \leq \gamma_j, \quad \forall\, j \in B \cup NB, \\
& \gamma_j = w_j y_j, \text{ if } i \in B, \\
& \gamma_j = w_j, \text{ if } i \in NB, \\
& \lambda^1, \lambda^2 \in \mathbb{R}^n_+, \mu_j \in \mathbb{R}^n, \forall j \in B \cup NB,\ \gamma \in \mathbb{R}^n.
\end{aligned}
$$

New bilevel programming approaches to the location of controversial facilities

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