Minmax Regret Maximal Covering on Networks with Edge Demands

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Outline

Introduction

- Single-facility location problem on a network.

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- Our aim is to maximise the covered demand.

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- Our aim is to maximise the covered demand.
- The demand distributed along the edges.
- The demand is uncertain with only a known interval estimation.

Introduction

Objective

Given:

- $G = (V, E)$: an undirected network; $|V| = n$ and $|E| = m$.
- R : coverage radius.
- For $e \in E$, two non-negative continuous functions $lb_e : [0,1] \to \mathbb{R}^+$ and $ub_e : [0,1] \to \mathbb{R}^+$ that specify the minimal and maximal demand along the edge.

The aim is to locate a facility along the network that minimise the *worst-case* of coverage loss.

Criteria

- Coverage criterion: Maximal Covering.

R. Church and C. ReVelle.
The maximal covering location problem.
Papers of the Regional Science Association, 3:101–118, 1974.

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  Let $x \in G$ be a facility:
  - $z \in G$ is *covered* by x, if $d(x, z) \leq R$.
  - $C(x) := \{z \in G \mid d(x, z) \leq R\}$ is the *coverage area* of x.

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  The covered demand on an edge $e \in E$ by x for a specific demand realisation w:

$$g_e(x, w) = \int_{y=(e,t)\in C_e(x)} w_e(t)\, dt. \tag{1}$$

  The total amount of covered demand on the network:

$$g(x, w) = \sum_{e \in E} g_e(x, w)\,. \tag{2}$$

Criteria

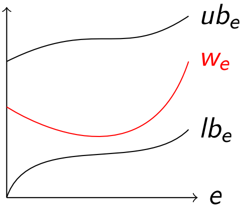
- Coverage criterion: Maximal Covering.

$$\max_{x \in G} g(x) = \sum_{e \in E} \int_{y=(e,t) \in C_e(x)} w_e(t)\, dt. \qquad (1)$$

O. Berman, J. Kalcsics, and D. Krass.
On covering location problems on networks with edge demand.
Computers & Operations Research, 74:214–227, 2016.

Criteria

- Coverage criterion: Maximal Covering.
- Uncertainty Demand: MinMax Regret. Minimise the *worst-case* of coverage loss.



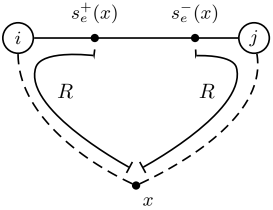
$$r^* = \min_{x \in X} \max_{lb \leq w \leq ub} \left(\max_{y \in G} g(y, w) - g(x, w) \right).$$

Edge coverage functions

If $x \notin e = [i, j]$

$$s_e^+(x) = \min\left\{1, \max\left\{0, \frac{R - d(x,i)}{\ell_e}\right\}\right\}$$

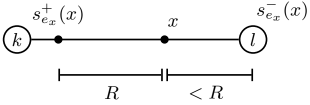
$$s_e^-(x) = \max\left\{0, \min\left\{1, 1 - \frac{R - d(x,j)}{\ell_e}\right\}\right\}$$



If $x \in e_x = [k, l]$

$$s_{e_x}^+(x) = \max\left\{0, \frac{d(x,k) - R}{\ell_{e_x}}\right\}$$

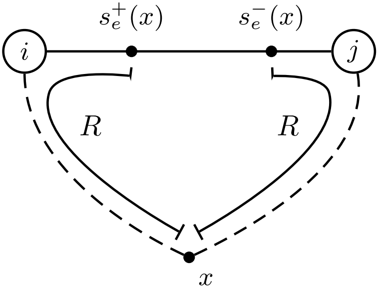
$$s_{e_x}^-(x) = \min\left\{1, 1 - \frac{d(x,l) - R}{\ell_{e_x}}\right\}$$



Coverage area $C_e(x)$ of partially covered edges

For $e \in E \setminus \{e_x\}$

$$C_e(x) = (e, [0, s_e^+(x)] \cup [s_e^-(x), 1]), \text{ if } 0 < s_e^+(x) < s_e^-(x) < 1$$
$$E^p(x) = E \setminus \{E^c(x) \cup E^u(x) \cup \{e_x\}\}.$$

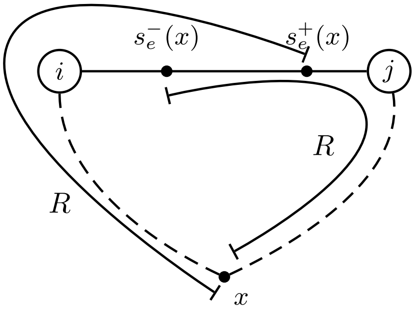


Coverage area $C_e(x)$ of covered edges

For $e \in E \setminus \{e_x\}$

$$C_e(x) = e, \text{ if } s_e^-(x) \leq s_e^+(x),$$

$$E^c(x) = \{e \in E \setminus \{e_x\} \mid s_e^-(x) \leq s_e^+(x)\},$$

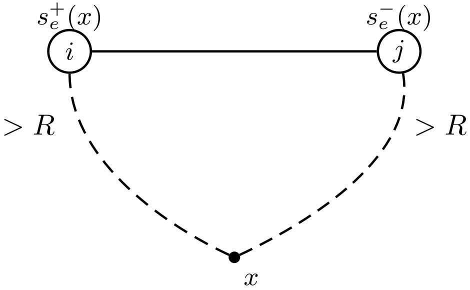


Coverage area $C_e(x)$ of uncovered edges

For $e \in E \setminus \{e_x\}$

$$C_e(x) = \emptyset, \text{ if } s_e^+(x) = 0, s_e^-(x) = 1,$$

$$E^u(x) = \{e \in E \setminus \{e_x\} \mid s_e^+(x) = 0 \text{ and } s_e^-(x) = 1\},$$



Coverage area

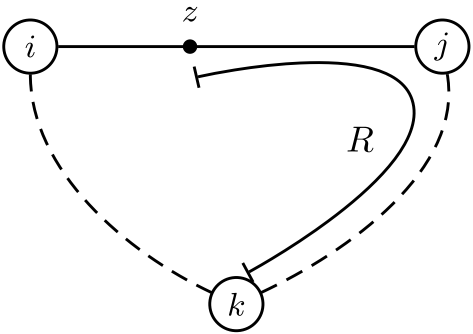
The total coverage can now be written as

$$g(x,w) = \int_{s_{e_x}^{+}(x)}^{s_{e_x}^{-}(x)} w_{e_x}(u)\,du + \sum_{e\in E^c(x)} \int_0^1 w_e(u)\,du + \sum_{e\in E^p(x)} \left(\int_0^{s_e^{+}(x)} w_e(u)\,du + \int_{s_e^{-}(x)}^1 w_e(u)\,du \right).$$

Singularity points (PP)

- Network intersect points $NP := \bigcup_{e \in E} NP_e$

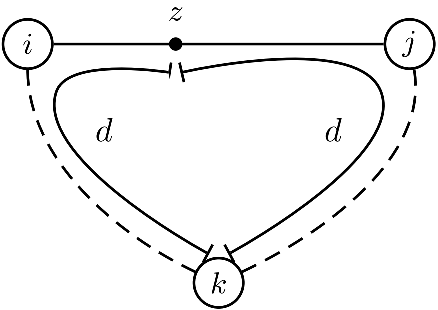
$$NP_e := \{z \in e \mid \exists\, k \in V : d(z,k) = R\};$$



Singularity points (PP)

- Bottleneck points $BP := \bigcup_{e \in E} BP_e$

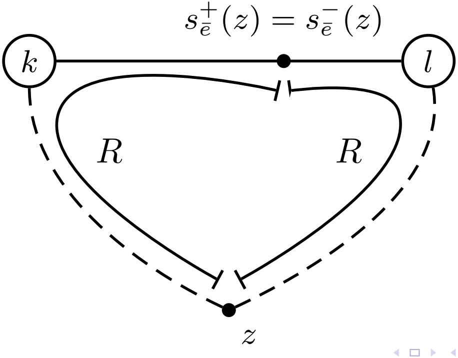
  $BP_e := \{z \in e \mid \exists k \in V : t\ell_e + d(k,i) = (1-t)\ell_e + d(k,j)\}$
  $e = [i,j]$



Singularity points (PP)

- Exact coverage point $EP := \bigcup_{e \in E} EP_e$

$$EP_e := \{z \in e \mid \exists\, \bar{e} \in E :\ 0 < s^+_{\bar{e}}(z) = s^-_{\bar{e}}(z) < 1\};$$



Singularity points (PP)

- $PP_e := \{i, j\} \cup NP_e \cup BP_e \cup EP_e$

  There are at most $\mathcal{O}(m)$ partition points on each $e \in E$.
- $PP := \bigcup_{e \in E} PP_e$

  There are $\mathcal{O}(m^2)$ on the whole network.

Lemma

$s_e^+(x)$ and $s_e^-(x)$, $e \in E$, are continuous and piecewise linear functions over $x \in e_x$ with a constant number of pieces.

Edge coverage functions and singularity points

Theorem

Let $e_x \in E, x \in [z^1, z^2]$ *such that* $z^1, z^2 \in PP_{e_x}$.

1. *The sets* $E^c(x)$, $E^u(x)$, *and* $E^p(x)$ *are identical for* $x \in [z^1, z^2]$.
2. $s_e^+(x)$ *and* $s_e^-(x)$ *have a unique linear representation for* $x \in [z^1, z^2]$.
3. $g_e(x, w)$, $e \in E$, *have a unique representation, for* $x \in [z^1, z^2]$ *and* w *a non-negative continuous demand function.*

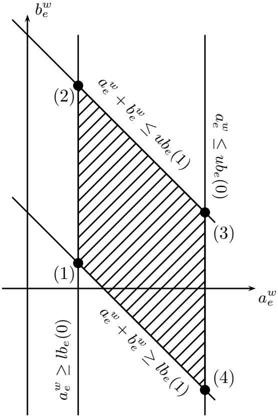
Linear Demand realisation

Let $lb_e(t) = a_e^{lb} + b_e^{lb} \cdot t$, $ub_e(t) = a_e^{ub} + b_e^{ub} \cdot t$, and $w_e(t) = a_e^{w} + b_e^{w} \cdot t$.

Linear Demand realisation

Let $lb_e(t) = a_e^{lb} + b_e^{lb} \cdot t$, $ub_e(t) = a_e^{ub} + b_e^{ub} \cdot t$, and $w_e(t) = a_e^w + b_e^w \cdot t$.

$$
\begin{aligned}
a_e^w &\geq lb_e(0) = a_e^{lb} \\
a_e^w &\leq ub_e(0) = a_e^{ub} \\
a_e^w + b_e^w &\leq ub_e(1) = a_e^{ub} + b_e^{ub} \\
a_e^w + b_e^w &\geq lb_e(1) = a_e^{lb} + b_e^{lb} \\
a_e^w &\geq 0,\ b_e^w \in \mathbb{R}
\end{aligned}
$$



Worst-case demand realisation

Theorem

The worst-case demand realisation for a fixed x, y, and e such that $x, y \in G$ and $e \in E$ can be obtained by solving the following linear program:

$$r_e(x,y) = \max_{lb_e \leq w_e \leq ub_e} a_e^w \left(c_e(y) - c_e(x)\right) + \frac{1}{2} b_e^w \left(\bar{c}_e^2(y) - \bar{c}_e^2(x)\right) \quad (1)$$

$$c_e(x) = \begin{cases} 1, & \text{if } e \in E^c(x), \\ 1 - (s_e^-(x) - s_e^+(x)), & \text{if } e \in E^p(x), \\ 0, & \text{if } e \in E^u(x), \\ s_{e_x}^-(x) - s_{e_x}^+(x), & \text{if } e = e_x, \end{cases}$$

$$\bar{c}_e^2(x) = \begin{cases} 1, & \text{if } e \in E^c(x), \\ 1 - ((s_e^-(x))^2 - (s_e^+(x))^2), & \text{if } e \in E^p(x), \\ 0, & \text{if } e \in E^u(x), \\ (s_{e_x}^-(x))^2 - (s_{e_x}^+(x))^2, & \text{if } e = e_x. \end{cases}$$

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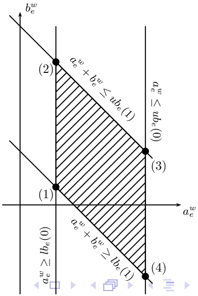
1. (a_e^{lb}, b_e^{lb}),
2. $(a_e^{lb}, a_e^{ub} + b_e^{ub} - a_e^{lb})$,
3. (a_e^{ub}, b_e^{ub}),
4. $(a_e^{ub}, a_e^{lb} + b_e^{lb} - a_e^{ub})$.



Worst-case demand realisation

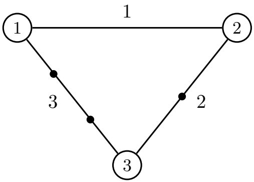
Theorem

An optimal solution of (1), (a_e^{w*}, b_e^{w*}), *is given by the first column of the following table whenever the conditions of columns 2-4 are fulfilled.*

(a_e^{w*}, b_e^{w*})	Conditions		
	$c_e(y) - c_e(x)$	$\bar{c}_e^2(y) - \bar{c}_e^2(x)$	$(\bar{c}_e^2(y) - \bar{c}_e^2(x)) - 2(c_e(y) - c_e(x))$
(a_e^{lb}, b_e^{lb})	≤ 0	≤ 0	≥ 0
$(a_e^{lb}, a_e^{ub} + b_e^{ub} - a_e^{lb})$	≥ 0	≥ 0	≥ 0
	≤ 0	≥ 0	$-$
(a_e^{ub}, b_e^{ub})	≥ 0	≥ 0	≤ 0
$(a_e^{ub}, a_e^{lb} + b_e^{lb} - a_e^{ub})$	≤ 0	≤ 0	≤ 0
	≥ 0	≤ 0	$-$

Algorithm

- Determine the set PP of partition points.



Let $R = 1$,

$$lb_{[1,2]}(t) = 3 - 3t, \quad ub_{[1,2]}(t) = 15 + 7t,$$
$$lb_{[2,3]}(t) = 6 - t, \quad ub_{[2,3]}(t) = 12 + 9t,$$
$$lb_{[1,3]}(t) = 2 + 3t, \quad ub_{[1,3]}(t) = 8 + 10t.$$

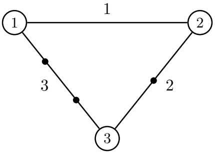
Algorithm

For $e = [i, j] \in E$:

- Sort the partition $PP_e = \{z^1, \ldots, z^{|PP_e|-1}\}$ in non-decreasing distance from node i.
- Derive the representation of the edge coverage functions over each sub-edge $[z^q, z^{q+1}]$, for $q \in \mathcal{I}_e := \{1, \ldots, |PP_e|-1\}$.

For $x_1 \in ([1,2], t_1)$ the edge coverage functions are given by

$$s^+_{[1,2]}(x_1) = 0, \qquad s^-_{[1,2]}(x_1) = 1,$$
$$s^+_{[2,3]}(x_1) = \frac{t_1}{2}, \qquad s^-_{[2,3]}(x_1) = 1,$$
$$s^+_{[1,3]}(x_1) = \frac{1-t_1}{3}, \qquad s^-_{[1,3]}(x_1) = 1.$$



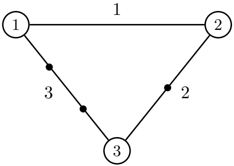
Algorithm

For $x_2 \in ([2,3], t_2)$ the edge coverage functions are given by

$$s^{+}_{[1,2]}(x_2) = 0, \qquad s^{-}_{[1,2]}(x_2) = \begin{cases} 2t_2, & \text{if } t_2 \leq \frac{1}{2} \\ 1, & \text{o/w} \end{cases},$$

$$s^{+}_{[2,3]}(x_2) = \begin{cases} \frac{2t_2-1}{2}, & \text{if } t_2 \geq \frac{1}{2} \\ 0, & \text{o/w} \end{cases}, \qquad s^{-}_{[2,3]}(x_2) = \begin{cases} \frac{1+2t_2}{2}, & \text{if } t_2 \leq \frac{1}{2} \\ 1, & \text{o/w} \end{cases},$$

$$s^{+}_{[1,3]}(x_2) = 0, \qquad s^{-}_{[1,3]}(x_2) = \begin{cases} \frac{4-2t_2}{3}, & \text{if } t_2 \geq \frac{1}{2} \\ 1, & \text{o/w} \end{cases}.$$



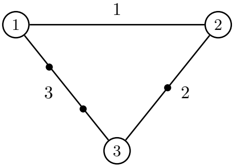
Algorithm

For $x_3 \in ([1,3], t_3)$ the edge coverage functions are given by

$$s^+_{[1,2]}(x_3) = \begin{cases} 1 - 3t_3 & \text{if } t_3 \le \frac{1}{3} \\ 0 & \text{o/w} \end{cases}, \quad s^-_{[1,2]}(x_3) = 1,$$

$$s^+_{[2,3]}(x_3) = 0, \qquad s^-_{[2,3]}(x_3) = \begin{cases} \frac{4-3t_3}{2} & \text{if } t_3 \ge \frac{2}{3} \\ 1 & \text{o/w} \end{cases},$$

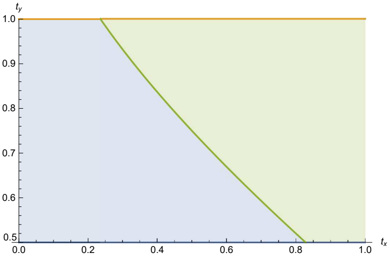
$$s^+_{[1,3]}(x_2) = \begin{cases} \frac{3t_3-1}{3} & \text{if } t_3 \ge \frac{1}{3} \\ 0 & \text{o/w} \end{cases}, \quad s^-_{[1,3]}(x_3) = \begin{cases} \frac{3t_3+1}{3} & \text{if } t_3 \le \frac{2}{3} \\ 1 & \text{o/w} \end{cases}.$$



Algorithm

for $e_x \in E$ **do**
  for $i \in \mathcal{I}_{e_x}$ **do**
    for $e_y \in E$ **do**
      for $j \in \mathcal{I}_{e_y}$ **do**
        Generate the subdivision in the rectangle $[z^i, z^{i+1}] \times [z^j, z^{j+1}]$ by the arcs defining the conditions of the worst case demand realisation for any $e \in E$.

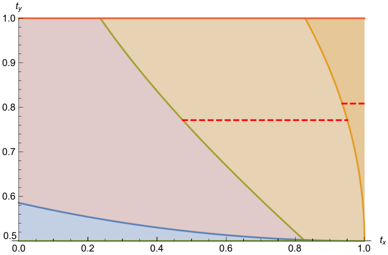
Let $e = [2,3], x = ([1,2], t_x)$, $y = ([2,3], t_y), \frac{1}{2} \leq t_y \leq 1$.
Cells for $r_{[2,3]}(x, y)$:



Algorithm

for $e_x \in E$ **do**
  for $i \in \mathcal{I}_{e_x}$ **do**
    for $e_y \in E$ **do**
      for $j \in \mathcal{I}_{e_y}$ **do**
        Let $\mathcal{C}^{ij}_{e_x e_y}$ be the family of arcs defined by:
        -boundaries of the cells previously obtained.
        -For any cell where $r(x, y)$ is concave, the intersection of the curve $\frac{\partial r}{\partial y}(x, y) = 0$ with that cell.

Let $x = ([1, 2], t_x)$ and $y = ([2, 3], t_y)$, $\frac{1}{2} \leq t_y \leq 1$. Cells for $r(x, y)$,



Algorithm

for $e_x \in E$ **do**
    for $i \in \mathcal{I}_{e_x}$ **do**
        Obtain the upper envelope, $h^i_{e_x}(x)$, of $r(x, y(x))$ of the arcs contained in $\displaystyle\bigcup_{e_y \in E, j \in \mathcal{I}_{e_y}} \mathcal{C}^{ij}_{e_x e_y}$.
        Find the minimum x^*_i of $h^i_{e_x}(x)$ over $[z^i, z^{i+1}]$.
        if $h^i_{e_x}(x^*_i) < r(x^*)$ **then**
            set $x^* := x^*_i$, $r(x^*) = h^i_{e_x}(x^*_i)$.

Algorithm

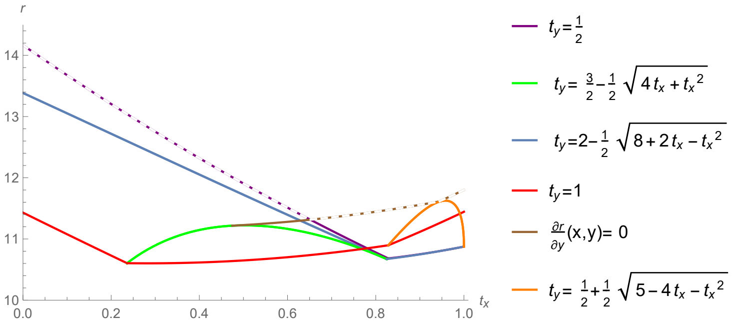


Figure: $r(x, y(x))$, $x = ([1,2], t_x)$ and $y = ([2,3], t_y)$, $\frac{1}{2} \leq t_y \leq 1$.

Algorithm

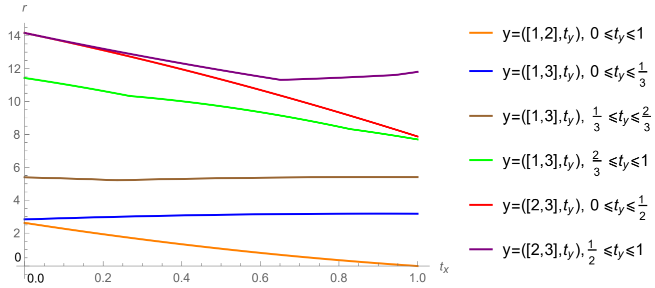


Figure: Upper envelope of $r(x, y(x))$, $x = ([1, 2], t_x)$.

The minimum value of r is 11.3153, where $x^*_{[1,2]} = ([1, 2], 0.6517)$. This should be repeated for each $x \in [z^i, z^{i+1}]$, where $i \in \mathcal{I}_{e_x}$ and $e_x \in E$.

Algorithm

Theorem

The single facility MinMax Regret Maximal Covering Location Problem on a network with edge linear demand realisations can be solved exactly in $\mathcal{O}(m^2\lambda_{10}(m^4))$ *time using the previous algorithm, where* $\lambda_s(m) = \max\{|U| : U \text{ is a } DS(m,s) - \text{sequence}\}$.

Theorem

The single facility MinMax Regret Maximal Covering Location Problem on a network with edge constant demand realisations can be solved exactly in $\mathcal{O}(m^2\lambda_3(m^4))$ *time using the previous algorithm, where* $\lambda_s(m) = \max\{|U| : U \text{ is a } DS(m,s) - \text{sequence}\}$.

Conclusions

- Although the majority of problems become NP-hard in the minmax regret version, we propose a polynomial time algorithm for solving the single-facility MinMax Regret Maximal Covering Location Problem on a network where the demand is
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- Although the majority of problems become NP-hard in the minmax regret version, we propose a polynomial time algorithm for solving the single-facility MinMax Regret Maximal Covering Location Problem on a network where the demand is
  - distributed along the edges,
  - constant or linear functions,
  - uncertain with only a known interval estimation.

Future work

Potential avenues for future research:

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- Other kind of demand realisation functions.
- Multi-facility location version of the problem.
- Apply a different criterion of coverage, e.g. the gradual covering.

Thanks for your attention