A Mixed Integer Linear Formulation for the Maximum Covering Location Problem with Ellipses

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Covering Problems

- Often facilities can serve only customers within a certain distance:
  - Emergency vehicles.
  - Wifi routers.
  - Mobile phones antennas.
- **Covering Problem:** locate some facilities such that:
  - Set Covering Problem: all customers are served at minimum cost, or
  - Maximal Coverage Problem: maximum number of customers is served with a budget limit.

Some History

- First mentions to covering problems in Berge (1957).
- First application: Police patroling in Hakimi (1965).
- First formulation:
  - Non-location context: Roth (1969).
  - Set covering problem: Toregas et al. (1971).
  - Maximal covering problem: Church and Revelle (1974).

Covering Geometry

- If Euclidean distance is used, coverage is determined by circles.
- Most of the problems studied in literature use circles.
- Other coverage distances: inclined parallelograms (Younies and Wessolowsky, 2004), block norms (Younies and Wesolowsky).
- Much less studied: ellipses.
- Wireless transmitter coverage: many satellite and antenna based transmitters coverage range has an elliptical shape.

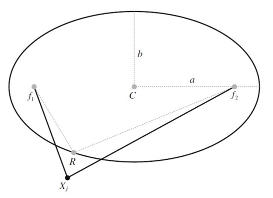
Example (Canbolat and von Massow, 2009)

Ellipse Geometry

- An ellipse with center C has two foci f_1 and f_2.
- It is the geometrical region of points R for which the sum of distances to the two foci is a constant value $2a$:

$$d_2(f_1, R) + d_2(f_2, R) = 2a.$$

- a is the semi-major axis.
- The values also determine a semi-minor axis b.
- Equation of the ellipse centered on (c_1, c_2): $\left(\frac{x-c_1}{a}\right)^2 + \left(\frac{y-c_2}{b}\right)^2 = 1.$



Context and Notation

- Problem on the plane.
- Demand points: $i = 1, \dots, n$.
- Demands w_i, $i = 1, \dots, n$.
- We locate k facilities with elliptical coverage out of a set of m.
- Each ellipse j has semi-major axis a_j and semi-minor axis b_j, $j = 1, \dots, m$.
- Each facility has a location cost h_j.
- Straight ellipses: both foci have the same vertical coordinate.

Canbolat and von Massow (2009) - Variables

- Decision variables:
  - x_{ij}: 1 if demand point i is covered by ellipse j, 0 otherwise.
  - y_j: 1 if ellipse j is chosen, 0 otherwise.
  - f_1x_j: x-coordinate for focus 1 of ellipse j.
  - f_1y_j: y-coordinate for focus 1 of ellipse j.
  - Notation: $f_{1j} = (f_1x_j, f_1y_j)$.
- Non-decision variables and data:
  - f_2x_j, f_2y_j: coordinates for focus 2.

$$f_2x_j = f_1x_j + 2a_j\sqrt{1 - \frac{b_j^2}{a_j^2}}, \quad f_2y_j = f_1y_j.$$

  - P_i: i-th demand point.

Canbolat and von Massow (2009) - Formulation

$$
\begin{aligned}
\text{Max.} \quad & \sum_{j=1}^{m}\sum_{i=1}^{n} w_i x_{ij} - \sum_{j=1}^{m} h_j y_j \\
\text{s.t.} \quad & d_2(f_{1j}, P_i) + d_2(f_{2j}, P_i) \leq 2a_j + M(1 - x_{ij}) \ \ \forall i, j, \\
& \sum_{j=1}^{m} y_j = k, \\
& x_{ij} \leq y_j \ \ \forall i, j, \\
& f_2x_j = f_1x_j + 2a_j\sqrt{1 - \frac{b_j^2}{a_j^2}} \ \ \forall j, \\
& f_2y_j = f_1y_j \ \ \forall j, \\
& \sum_{j=1}^{m} x_{ij} \leq 1 \ \ \forall i, \\
& y_j, x_{ij} \in \{0, 1\} \ \ \forall i, j, \\
& f_1x_j, f_1y_j, f_2x_j, f_2y_j \in \mathbb{R} \ \ \forall j.
\end{aligned}
$$

Canbolat and von Massow (2009) - Remarks

- Nonlinear model.
- Not really used by the authors.
- Instead they use a simulated annealing heuristic.
- We will not include them in our computational study.

Andretta and Birgin (2013) - Notation

- Same x_{ij} and y_j variables.
- $c_j = (c_j^x, c_j^y)$: coordinates of the center of ellipse j.
- $P_i = (p_i^x, p_i^y)$.

Andretta and Birgin (2013) - Formulation

$$
\begin{aligned}
\text{Max.} \quad & \sum_{j=1}^{m}\sum_{i=1}^{n} w_i x_{ij} - \sum_{j=1}^{m} h_j y_j \\
\text{s.t.} \quad & \sum_{j=1}^{m} y_j = k, \\
& x_{ij} \le y_j \quad \forall i,j, \\
& \sum_{j=1}^{m} x_{ij} \le 1 \quad \forall i, \\
& \left(\frac{c_j^x - p_i^x}{a_j}\right)^2 + \left(\frac{c_j^y - p_i^y}{b_j}\right)^2 \le 1 + M(1 - x_{ij}) \quad \forall i,j, \\
& y_j, x_{ij} \in \{0,1\} \quad \forall i,j, \\
& c_j^x, c_j^y \in \mathbb{R} \quad \forall j.
\end{aligned}
$$

Andretta and Birgin (2013) - Comments

- Similar model to Canbolat and van Massow's.
- Decision variables for centers instead of foci.
- Second-order programming model.
- CPLEX can solve moderate size instances.
- Andretta and Birgin propose exact and heuristic algorithms with a strong enumerative component.

Our Model - Key Result

- Given two points P_1 and P_2, and a fixed ellipse shape.
- Let us consider the intersection of the two ellipses centered on P_1 and P_2.
- The intersection is nonempty if and only if the intersection points are centers of ellipses that can cover both i and j.
- This results holds for any number of points.
- Helly's Tehorem: Given convex sets on the plane $S_1, S_2, \ldots, S_t$, if every triplet of sets has nonempty intersection, then the whole family has nonempty intersection.
- Translated to our problem: if three ellipses have empty intersection, then at most two points can be covered by the same ellipse.

Our Model - Key Result (2)

- For having easier shapes, we can consider circles with radius 1. For a given ellipse j:
  - The center is given by $\tilde{c}_j^x = c^x/a_j,\ \tilde{c}_j^y = c^y/b_j$.
  - Each point is given by $\tilde{p}_j^x = p^x/a_k,\ \tilde{p}_j^y = p^y/b_j$.
  - Instead of checking the ellipse equation

$$\left(\frac{c_j^x - p_i^x}{a_j}\right)^2 + \left(\frac{c_j^y - p_i^y}{b_j}\right)^2 \leq 1$$

  we check the circle equation

$$\left(\tilde{c}_j^x - \tilde{p}_i^x\right)^2 + \left(\tilde{c}_j^y - \tilde{p}_i^y\right)^2 \leq 1.$$

- We substitute the nonlinear constraints with empty triple intersection constraints:

$$x_{i_1 j} + x_{i_2 j} + x_{i_3 j} \leq 2 \quad \forall i_1, i_2, i_3 \ / \ D_{i_1}^j \cap D_{i_2}^j \cap D_{i_3}^j = \emptyset,$$

where D_i^j is the unit disk centered on $(p_i^x/a_j, p_i^y/b_j)$.

Our Model - Formulation 1

$$\text{Max.} \quad \sum_{j=1}^{m}\sum_{i=1}^{n} w_i x_{ij} - \sum_{j=1}^{m} h_j y_j$$

$$\text{s.t.} \quad \sum_{j=1}^{m} y_j = k,$$

$$x_{ij} \leq y_j \quad \forall i, j,$$

$$\sum_{j=1}^{m} x_{ij} \leq 1 \quad \forall i,$$

$$x_{i_1 j} + x_{i_2 j} + x_{i_3 j} \leq 2 \quad \forall i_1, i_2, i_3, j \ / \ D_{i_1}^{j} \cap D_{i_2}^{j} \cap D_{i_3}^{j} = \emptyset,$$

$$y_j, x_{ij} \in \{0, 1\} \quad \forall i, j,$$

Our Model - Remarks

- This a mixed integer linear model.
- The center variables are not included.
- But finding a the center of a circle of given radius that covers a given set of points can be done in linear time.
- Triple intersection constraints can be too many: $\mathcal{O}(mn^3)$, whereas all the other constraints are $1 + mn + n$.
- We can use a much smaller formulation based on a simple observation: if two sets have empty intersection, then they have empty intersection with any third set.

Our Model - Formulation 2

$$
\begin{aligned}
\text{Max.} \quad & \sum_{j=1}^{m}\sum_{i=1}^{n} w_i x_{ij} - \sum_{j=1}^{m} h_j y_j \\
\text{s.t.} \quad & \sum_{j=1}^{m} y_j = k, \\
& x_{ij} \le y_j \quad \forall i, j, \\
& \sum_{j=1}^{m} x_{ij} \le 1 \quad \forall i, \\
& x_{i_1 j} + x_{i2j} \le 1 \quad \forall i_1, i_2, j \ / \ D^j_{i_1} \cap D^j_{i_2} = \emptyset, \\
& x_{i_1 j} + x_{i_2 j} + x_{i_3 j} \le 2 \quad \forall i_1, i_2, i_3, j \ / \ D^j_{i_1} \cap D^j_{i_2} \cap D^j_{i_3} = \emptyset, \\
& D^j_{i_1} \cap D^j_{i_2} \ne \emptyset, \ D^j_{i_1} \cap D^j_{i_3} \ne \emptyset, \ D^j_{i_2} \cap D^j_{i_3} \ne \emptyset, \\
& y_j, x_{ij} \in \{0, 1\} \quad \forall i, j,
\end{aligned}
$$

Conditions

- Windows 7 64-bit Intel Core i5-3470, 2 cores 3.2GHz, 8GB RAM.
- CPLEX 12.8.
- C++ Concert Library.
- Visual Studio Community 2017.
- All the CPLEX tricks allowed.
- Time limit: 30 minutes.

Results for cm Instances

m: Number of demand points (all with demand 1).
n: Number of ellipses available.
k: Number of ellipses located.
z: Best lower bound.
G: Optimality gap (percentage).
T: Computational time in seconds.
$2i$: Number of double intersection constraints added.
$3i$: Number of triple intersection constraints added.

(m,n,k)	z	G	T	z	G	T	3i	z	G	T	BB	2i	3i
(25,3,1)	2.0	0	0	2.0	0	1	6771	2	0	0	1	750	6
(25,3,2)	3.8	0	1	3.8	0	28	6771	3.8	0	0	1	750	6
(25,3,3)	3.0	0	3	3.0	0	319	6771	3	0	0	1	750	6
(50,3,1)	4.2	0	1	4.2	0	23	57713	4.2	0	0	1	3086	56
(50,3,2)	8.2	0	4	8.0	241	1800	57713	8.2	0	1	1	3086	56
(50,3,3)	10.0	0	72	10.0	287	1800	57713	10	0	1	1	3086	56
(100,3,1)	12.2	0	2	12.2	269	1800	475614	12.2	0	2	1	12438	312
(100,3,2)	20.0	0	190	9.8	854	1800	475614	20	0	5	1	12438	312
(100,3,3)	27.0	42	1800	22.0	309	1800	475614	27	0	9	1	12438	312

Conclusions

- Ellipse covering is a problem not widely studied.
- The "default" formulation has quadratic big M constraints.
- Moderate size instance can be solved, but not large instances.
- A geometrical analysis of the problem allows us to give a linear formulation that can solve moderate size instances easily.
- We need to be careful with the size of that formulation.

Future Research

- Goal: To solve very large instances.
- How? For example:
  - Generate intersection constraints only when needed. Most are nonbinding.
  - Make good use of intersection constraints (redundance, domination, coefficient lifting).
- Include rotations.