Minimum Enclosing Polyellipses

Víctor Blanco
Universidad de Granada

(joint work with J. Puerto)

IX International Workshop on Locational Analysis and Related Problems
Cádiz, 2019

Outline

Weber Problem

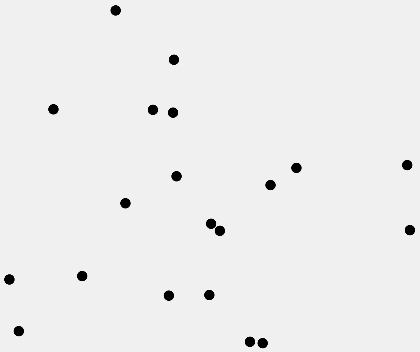
Given a finite set of demand points $\mathcal{U} \subseteq \mathbb{R}^d$, Continuous Facility Location Problems (CFLP) deal with the determination of optimal positions in such a space by minimizing certain measure of the distances to the points.

Weber Problem: $\displaystyle\min_{x\in\mathbb{R}^d} \sum_{u\in\mathcal{U}} \omega_u \|x - u\| \ (\omega_u > 0)$

Weber Problem

Given a finite set of demand points $\mathcal{U} \subseteq \mathbb{R}^d$, Continuous Facility Location Problems (CFLP) deal with the determination of optimal positions in such a space by minimizing certain measure of the distances to the points.

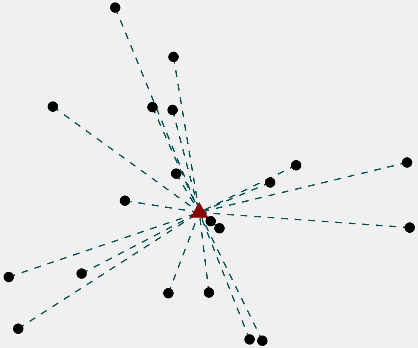
Weber Problem: $\min\limits_{x \in \mathbb{R}^d} \sum\limits_{u \in \mathcal{U}} \omega_u \|x - u\| \; (\omega_u > 0)$



Weber Problem

Given a finite set of demand points $\mathcal{U} \subseteq \mathbb{R}^d$, Continuous Facility Location Problems (CFLP) deal with the determination of optimal positions in such a space by minimizing certain measure of the distances to the points.

Weber Problem: $\displaystyle\min_{x \in \mathbb{R}^d} \sum_{u \in \mathcal{U}} \omega_u \|x - u\| \ (\omega_u > 0)$



Weber Problem

- The unweighted (resp. weighted) 3-points planar problem was posed by Fermat (1640) (resp. Simpson, 1750).
- Introduced by Weber (1909) in the context of facility location.
- For non collinear demand points, the problem is strictly convex, and then, it has an unique solution.
- Iterative popular procedure: **Weiszfeld's** algorithm (1937)
- Several extensions: different metrics, multiple facilities, ordered median aggregation of distances, barriers, dimensional facilities, etc

Polyellipses

Let us consider the planar Euclidean Weber problem whose objective function to minimize is:

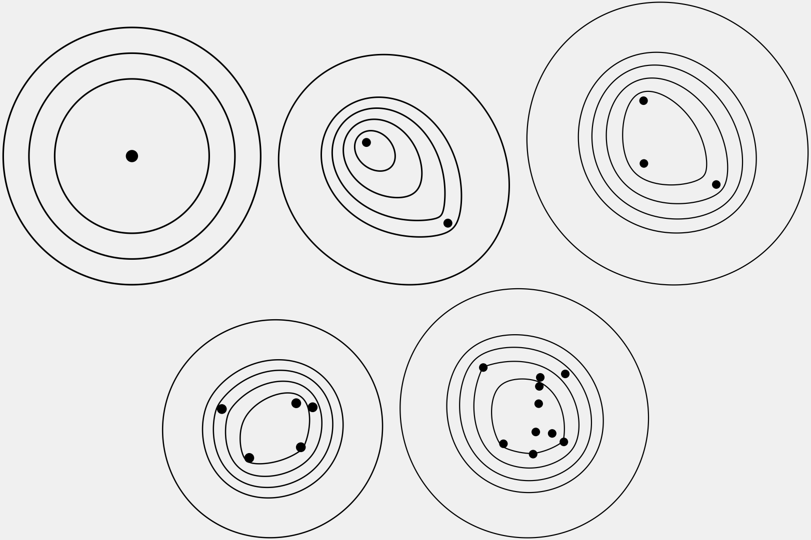
$$\Phi(x) = \sum_{u \in \mathcal{U}} \omega_u \|u - x\|_2, \text{ for } x \in \mathbb{R}^2$$

The levels curves are given by the following sets:

$$\mathbb{E}(\mathcal{U}, \omega, r) = \left\{ x \in \mathbb{R}^2 : \sum_{u \in \mathcal{U}} \omega_u \|u - x\|_2 = r \right\}$$

for $r \geq 0$.
$\mathbb{E}(\mathcal{U}, \omega, r)$ are called (weighted) **polyellipses** (a.k.a. multifocal ellipses, oval curves, $|\mathcal{U}|$-ellipses, ...).

Polyellipses



Polyellipses

$$\Phi(x) = \sum_{u \in \mathcal{U}} \omega_u ||u - x||_2, \text{ for } x \in \mathbb{R}^2$$

The levels curves are given by the following sets:

$$\mathbb{E}(\mathcal{U}, \omega, r) = \left\{ x \in \mathbb{R}^2 : \sum_{u \in \mathcal{U}} \omega_u ||u - x||_2 = r \right\}$$

for $r \geq 0$.
$\mathbb{E}(\mathcal{U}, \omega, r)$ are called (weighted) **polyellipses** (a.k.a. multifocal ellipses, oval curves, $|\mathcal{U}|$-ellipses, egglipses, ...) with foci $\mathcal{U}$ and radius r.

Polyellipses

$$\Phi(x) = \sum_{u \in \mathcal{U}} \omega_u \|u - x\|_2, \text{ for } x \in \mathbb{R}^2$$

The levels curves are given by the following sets:

$$\mathbb{E}(\mathcal{U}, \omega, r) = \left\{x \in \mathbb{R}^2 : \sum_{u \in \mathcal{U}} \omega_u \|u - x\|_2 = r\right\}$$

for $r \geq 0$.
$\mathbb{E}(\mathcal{U}, \omega, r)$ are called (weighted) **polyellipses** (a.k.a. multifocal ellipses, oval curves, $|\mathcal{U}|$-ellipses, egglipses, ...) with foci $\mathcal{U}$ and radius r.

- J.C. Maxwel (1851): Its first paper with 14 years analyzed oval ellipses.
- Melzak & Forsyth (1977): Rediscovering of Weiszfeld's Algorithm and drawing strategies.
- Ërdos & Vincze (1982), Vincze & Nagy (2010, 2011), : Approximation of convex bodies by polyellipses.
- Varga & Vincze (2008); Vinzce (2013): Differential Geometry on polyellipses.
- Nie, Parrilo, Sturmfels (2008): Semidefinite representation of polyellipses.
- Vincze et. al (2018): Generalization of Ërdos-Vincze approximation.

Polyellipses

Let us denote by $\mathbb{P}(\mathcal{U}, \omega, r)$ the region bounded by the polyellipse:

$$\mathbb{P}(\mathcal{U}, \omega, r) = \left\{ x \in \mathbb{R}^2 : \sum_{u \in \mathcal{U}} \omega_u ||x - u||_2 \leq r \right\}$$

Polyellipses

Let us denote by $\mathbb{P}(\mathcal{U}, \omega, r)$ the region bounded by the polyellipse:

$$\mathbb{P}(\mathcal{U}, \omega, r) = \left\{ x \in \mathbb{R}^2 : \sum_{u \in \mathcal{U}} \omega_u \|x - u\|_2 \leq r \right\}$$

- $\mathbb{P}(\mathcal{U}, \omega, r)$ is a closed, bounded and convex set.
- $\mathbb{P}(\mathcal{U}, \omega, r)$ is a SOC representable set.

Polyellipses

Let us denote by $\mathbb{P}(\mathcal{U}, \omega, r)$ the region bounded by the polyellipse:

$$\mathbb{P}(\mathcal{U}, \omega, r) = \left\{ x \in \mathbb{R}^2 : \sum_{u \in \mathcal{U}} \omega_u \|x - u\|_2 \leq r \right\}$$

- $\mathbb{P}(\mathcal{U}, \omega, r)$ is a closed, bounded and convex set.
- $\mathbb{P}(\mathcal{U}, \omega, r)$ is a SOC representable set.

Let $\mathcal{U} \subseteq \mathbb{R}^d$ a finite set of non collinear points and $x^* \in \mathbb{R}^d$ be the solution to the Weber problem $r^* = \min_{x \in \mathbb{R}^2} \sum_{u \in \mathcal{U}} \omega_u \|u - \mathbf{x}\|_2$, then:

1. $\mathbb{P}(\mathcal{U}, \omega, r^*) = \{x^*\}$.
2. $\mathbb{P}(\mathcal{U}, \omega, r) = \emptyset$ for all $r < r^*$.
3. If $\mathbb{P}(\mathcal{U}, \omega, r)$ has nonempty interior for every $r > r^*$.
4. $x^* \in \mathbb{P}(\mathcal{U}, \omega, r)$ for all $r \geq r^*$.

(Nie, Parrilo, Sturmfels; 2008): $\mathbb{P}(\mathcal{U}, \omega, r)$ can be represented as the set of solutions of a polynomial equation with degree $2|\mathcal{U}|$ (if $|\mathcal{U}|$ is odd) or $2^{|\mathcal{U}|}$ - $\binom{|\mathcal{U}|}{\frac{|\mathcal{U}|}{2}}$ (if $|\mathcal{U}|$ is even) which can be expressed as the determinant of a symmetric matrix of linear polynomials.

Minimum Enclosing Polyellipse

Given:

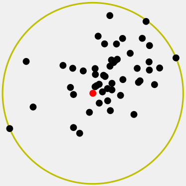
- A finite set of demand points $\mathcal{A} \subset \mathbb{R}^2$.
- A finite set of foci $\mathcal{U} \subseteq \mathbb{R}^2$ (noncollinear)
- Weights ω_u, $\forall u \in \mathcal{U}$.

Find:

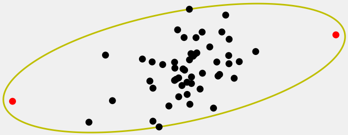
- Radius $r \geq 0$,
- Translation $x \in \mathbb{R}^2$

such that $\mathcal{A} \subseteq \mathbb{P}(\mathcal{U} + x, \omega, r)$ with minimum r.

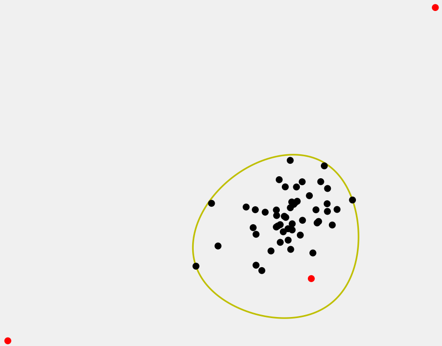
Minimum Enclosing Polyellipse



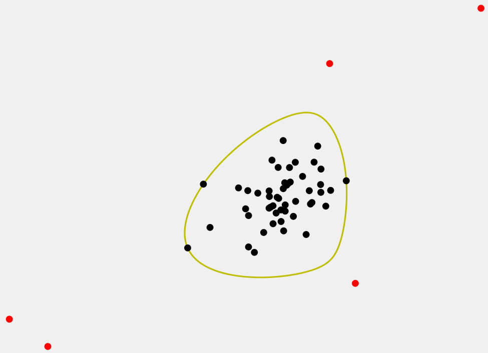
Minimum Enclosing Polyellipse



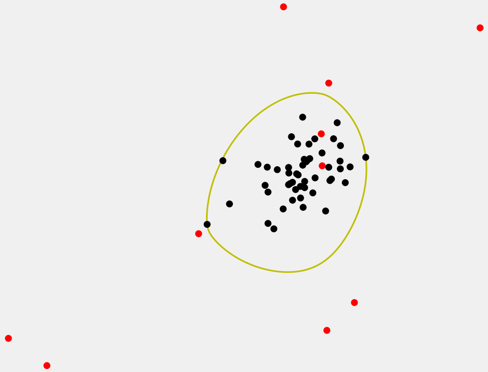
Minimum Enclosing Polyellipse



Minimum Enclosing Polyellipse



Minimum Enclosing Polyellipse



Minimum Enclosing Polyellipse

- For $|\mathcal{U}| = 1$, MEP $=$ **1-center** problem.
- The problem can be formulated as:

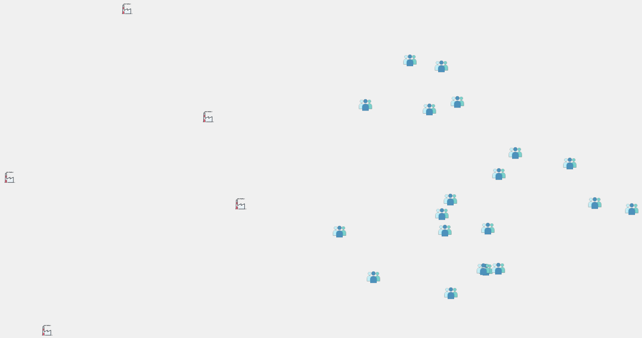
$$\min_{x \in \mathbb{R}^2} \max_{a \in \mathcal{A}} \sum_{u \in \mathcal{U}} \omega_u ||a - u - x||$$

or equivalently:

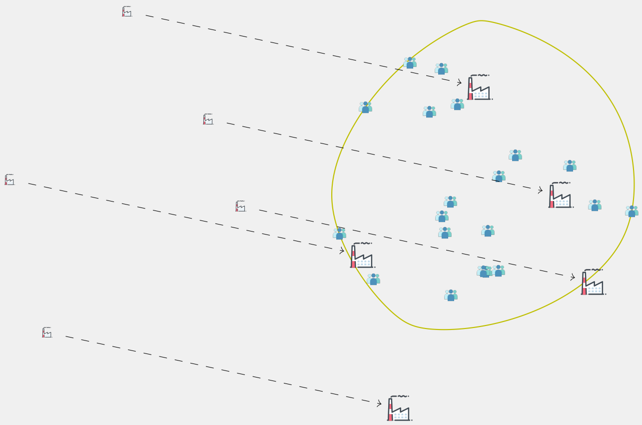
$$\begin{aligned} &\min_{x \in \mathbb{R}^d} \; r \\ &\text{s.t.} \; \sum_{u \in \mathcal{U}} \omega_u ||a - u - x|| \leq r, \forall a \in \mathcal{A} \end{aligned}$$

- The problem can be seen as the location of $|\mathcal{U}|$ facilities with constant relative distances $(\mathcal{U} + x)$, such that the maximum cost from a demand point to all the facilities is minimized.

Minimum Enclosing Polyellipse



Minimum Enclosing Polyellipse



SOCP

$$
\begin{aligned}
\min\ & r & \\
\text{s.t.}\ & \sum_{u \in \mathcal{U}} \omega_u d_{au} \leq r, & \forall a \in \mathcal{A} \\
& ||a - u - x|| \leq d_{au}, & \forall u \in \mathcal{U}, a \in \mathcal{A} \\
& x \in \mathbb{R}^2, r \in \mathbb{R}, & \\
& d_{au} \in \mathbb{R}^{|\mathcal{U}|}, & \forall u \in \mathcal{U}, a \in \mathcal{A}.
\end{aligned}
$$

Poly-time solvable by i.p. methods:
(**Nesterov & Todds, 1997**): For a given accuracy $\epsilon > 0$, the problem can be solved in $O\left(|\mathcal{A}||\mathcal{U}|^{3.5} \log(\frac{1}{\epsilon})\right)$-time.

Dual Approach

Let $f_a(x) = \sum_{u \in \mathcal{U}} \omega_u ||a - u - x||$, for $a \in A$, i.e., the distance/cost supported by each single customer to be serviced by all the facilities/foci.

$$r^* = \min_{x \in \mathbb{R}^d} \; r$$
$$\text{s.t. } f_a(x) \leq r, \forall a \in \mathcal{A}$$

Dual Approach

Let $f_a(x) = \sum_{u \in \mathcal{U}} \omega_u ||a - u - x||$, for $a \in A$, i.e., the distance/cost supported by each single customer to be serviced by all the facilities/foci.

$$\begin{aligned} r^* = \min_{x \in \mathbb{R}^d} \; & r \\ \text{s.t.} \; & f_a(x) \leq r, \forall a \in \mathcal{A} \end{aligned}$$

Define:

$$F(\alpha) = \min_x \sum_{a \in \mathcal{A}} \alpha_a f_a(x) = \min_x \sum_{a \in \mathcal{A}} \alpha_a \sum_{u \in \mathcal{U}} \omega_u ||a - u - x||,$$

Dual Approach

Let $f_a(x) = \sum_{u \in \mathcal{U}} \omega_u \|a - u - x\|$, for $a \in A$, i.e., the distance/cost supported by each single customer to be serviced by all the facilities/foci.

$$
\begin{aligned}
r^* = \min_{x \in \mathbb{R}^d} \quad & r \\
\text{s.t.} \quad & f_a(x) \leq r, \forall a \in \mathcal{A}
\end{aligned}
$$

Define:

$$F(\alpha) = \min_x \sum_{a \in \mathcal{A}} \alpha_a f_a(x) = \min_x \sum_{a \in \mathcal{A}} \alpha_a \sum_{u \in \mathcal{U}} \omega_u \|a - u - x\|,$$

Then:

$$r^* = \max_{\alpha \in \Delta_{\mathcal{A}}} \quad F(\alpha)$$

$$\text{where } \Delta_{\mathcal{A}} = \{\alpha \in \mathbb{R}_+^{|\mathcal{A}|} : \sum_{a \in \mathcal{A}} \alpha_a = 1\}$$

Projected Gradient Descent approach

$$\frac{\partial F}{\partial \alpha_a}(\alpha) = f_a(x^*_\alpha) = \sum_{u \in \mathcal{U}} \omega_u ||a - u - x^*_\alpha||, \quad \forall a \in \mathcal{A}.$$

With such an information, a projected gradient descent method can be applied to solve the dual problem. Let start with an initial solution $\alpha^0 \in \Delta$ and iterate:

$$\alpha^{k+1} = \Pi_\Delta(\alpha^k - \eta^k \nabla_\alpha F(\alpha^k)).$$

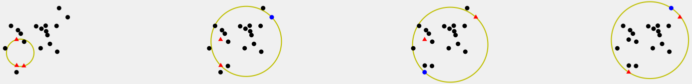
where $\Pi_\Delta(\beta) = \arg\min_{\gamma \in \Delta} ||\beta - \gamma||$ is the orthogonal projection onto the unit simplex which can be efficiently (Chen & Ye, 2011; Wang & Carreira-Perpiñan, 2013) in $O(n \log n)$.

Elzinga-Hearn Approach

(**Elzinga & Hearn, 1972**) found an $O(|\mathcal{A}|^2)$ algorithm for finding the smallest enclosing circle:

Algorithm 1: EH

Data: $\mathcal{A}$ and $A^0 \subseteq \mathcal{A}$ with $|A^0| = 3$. $k = 0$
Result: r^k and x^k
1. Compute the minimum radius circle covering A^k: r^k, x^k.
2. **if** *All the points covered* **then**
    STOP
else
    $A^{k+1} = A^k \cup \{a^k\} \setminus \{b^k\}$.
    $k = k + 1$
end



Also: $O(n \log n)$ (Farthest Voronoi Diagrams - **Shamos & Hoey, 1975**), $O(n)$ (**Meggido, 1983**, **Welz, 1991**).

Elzinga-Hearn Approach

$$\mathcal{C}_a(r^*) = \{z \in \mathbb{R}^d : \sum_{u \in \mathcal{U}} ||a - u - z|| \leq r^*\}, \forall a \in \mathcal{A},$$

(which is a polyellipse with foci $\{a - u : u \in \mathcal{U}\}$ and radius r^*. Then, $x^* \in \bigcap_{a \in \mathcal{A}} \mathcal{C}_a$. Actually, since r^* is minimum:

$$\bigcap_{a \in \mathcal{A}} \mathcal{C}_a(r^*) = \{x^*\}$$

By **Helly's Theorem**: There exist $\mathcal{A}' \subseteq \mathcal{A}$ with $|\mathcal{A}'| = 3$ such that $\bigcap_{a \in \mathcal{A}'} \mathcal{C}_a(r^*) = \{x^*\}$.

Elzinga-Hearn Approach

Algorithm 2: EH for polyellipses

Data: $\mathcal{A}$ and $A^0 \subseteq \mathcal{A}$ with $|A^0| = 3$. $k = 0$. $\mathcal{U}$

Result: r^k and x^k

1. Compute the minimum radius polyellipse covering A^k: r^k, x^k.

2. Let $\rho^k = \max_{a \in \mathcal{A}} \sum_{u \in \mathcal{U}} \omega_u \|a - u - x^*\|$ and $a^k \in \mathcal{A}$ reaching such a maximum.

if $\rho^k = r^k$ **then**

  STOP

else

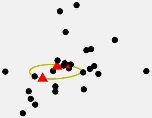
  $A^{k+1} = A^k \cup \{a^k\} \setminus \{b^k\}$ with b^k is the element in A^k such that its substitution from the set by a^k reaches the maximum value of the objective function.

  $k = k + 1$.

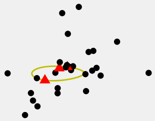
end

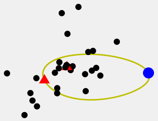
(**Nesterov & Todds, 1997**): For a given accuracy $\epsilon > 0$, EH can solve the problem in $O\left(|\mathcal{A}|^2 |\mathcal{U}|^{3.5} \log(\frac{1}{\epsilon})\right)$-time.

Elzinga-Hearn Approach

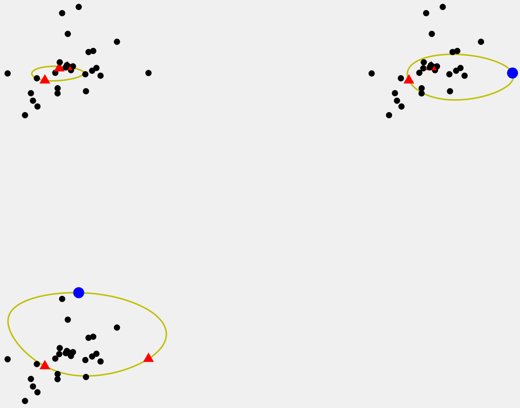


Elzinga-Hearn Approach

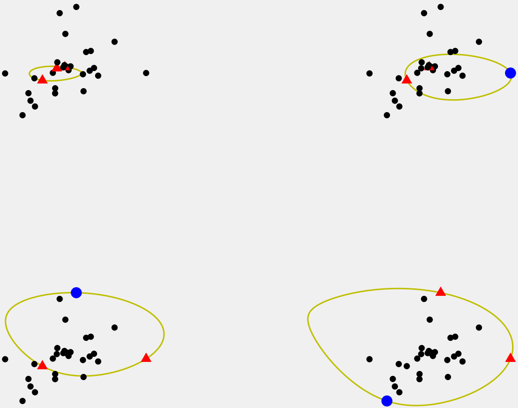




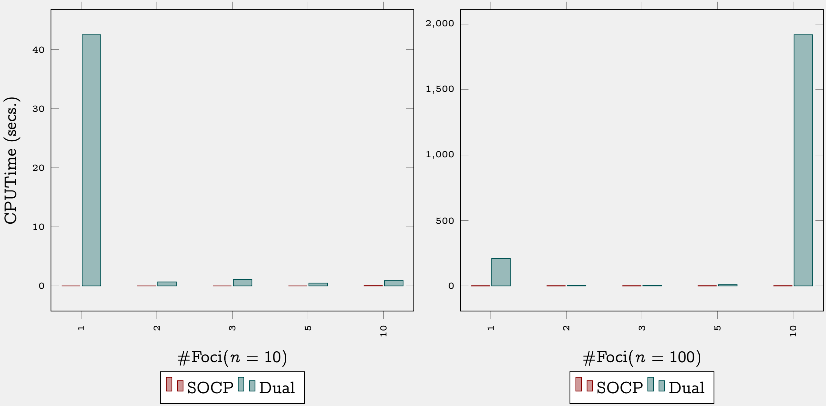
Elzinga-Hearn Approach



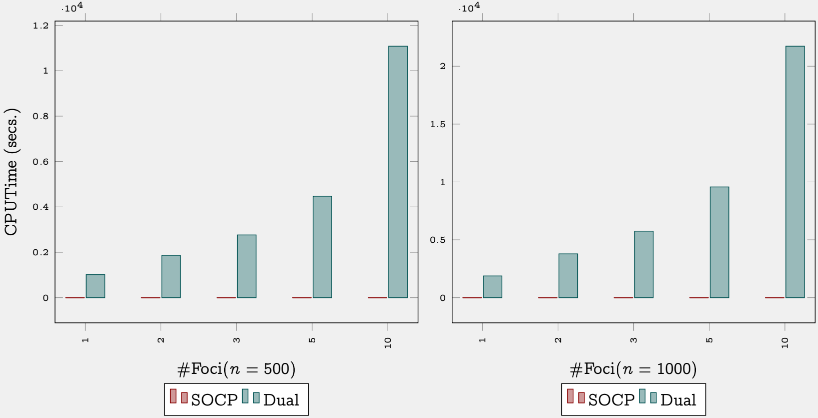
Elzinga-Hearn Approach



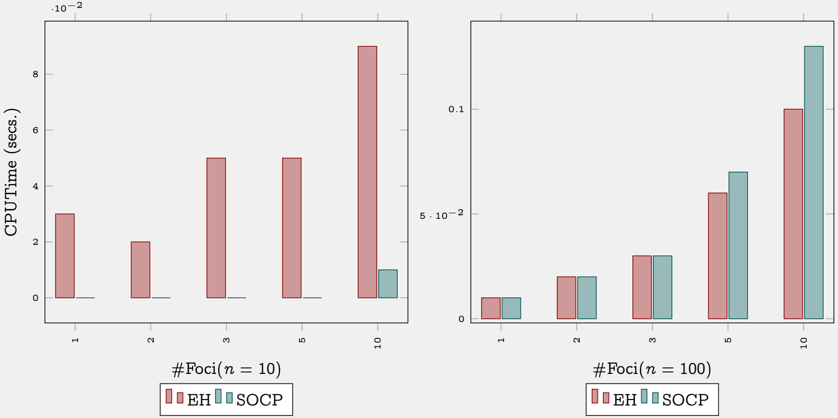
Experiments: SOCP vs. Dual



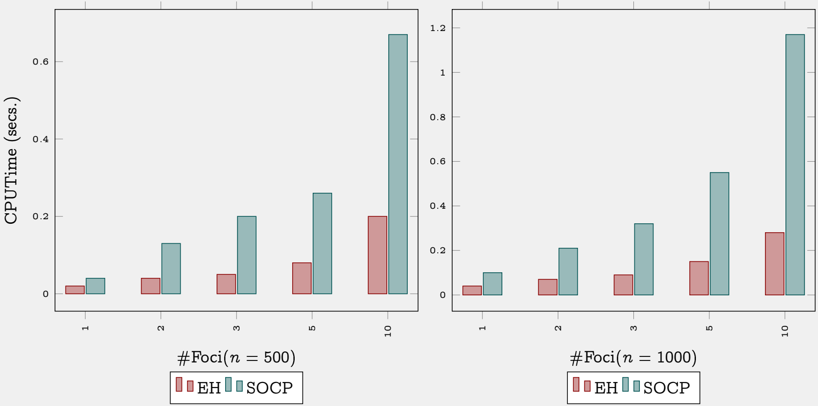
Experiments: SOCP vs. Dual



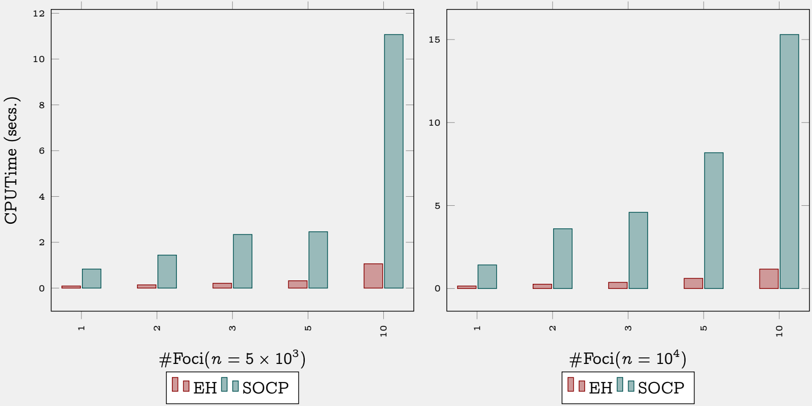
Experiments: SOCP vs. EH



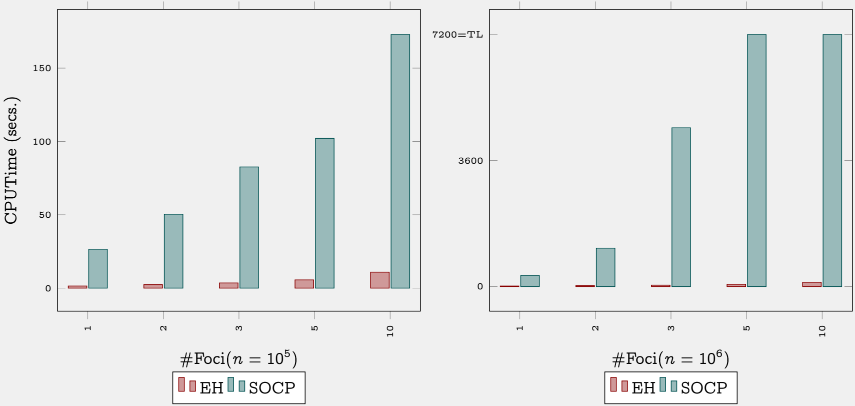
Experiments: SOCP vs. EH



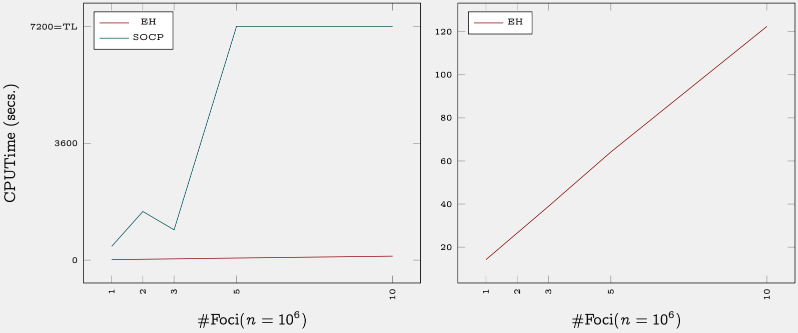
Experiments: SOCP vs. EH



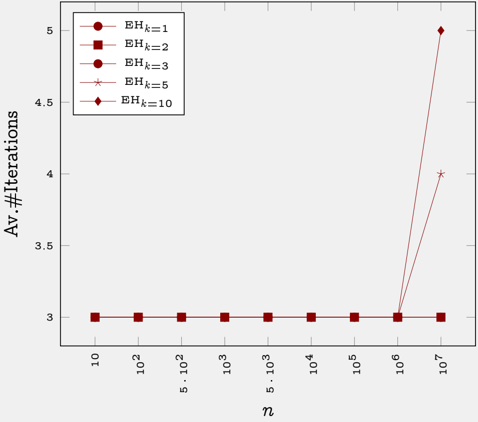
Experiments: SOCP vs. EH



Experiments: SOCP vs. EH



Experiments: EH (iterations)



Select the foci

Let $\mathcal{B}$ a potential set of foci.

$$y_u = \begin{cases} 1 & \text{if } u \text{ is selected as a foci} \\ 0 & \text{otherwise} \end{cases} \qquad u \in \mathcal{B}$$

The following model allows us to determine the p foci and the polyellipse

$$\begin{aligned} \min \quad & r \\ \text{s.t.} \quad & r \geq \sum_{u \in \mathcal{B}} d_{au}, \forall a \in \mathcal{A}, \\ & d_{au} \geq ||a - u - x|| - M_{au}(1 - y_u), \forall a \in \mathcal{A}, u \in \mathcal{B}, \\ & \sum_{u \in \mathcal{B}} y_u = p, \\ & r, d_{au} \geq 0, \\ & y_u \in \{0, 1\}. \end{aligned}$$

Select the foci: EH

Algorithm 3: EH-Select

Data: $\mathcal{A}$ and $A^0 \subseteq \mathcal{A}$ with $|A^0| = 3$. $k = 0$. $\mathcal{F} = \emptyset$, $UB = \infty$, $LB = 0$.
Result: r^k, x^k and selected foci
while $UB > LB$ **do**

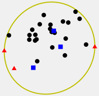
1. $\mathcal{U}^{k+1} = Foci_{\mathcal{F}}(I^k) \rightarrow LB_k$ ($LB = \max\{LB, LB_k\}$). If infeasible: STOP.
2. $I^{k+1} = EH(\mathcal{U}^{k+1}) \rightarrow UB_k$ ($UB = \min\{UB, UB_k\}$).
3. $\mathcal{F} = \mathcal{F} \cup \ \mathcal{U}^{k+1}$, $k \rightarrow k+1$.

end

where $Foci_{\mathcal{F}}(I)$ determines the **optimal** p foci to enclose the points in I:

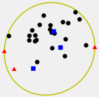
$$\begin{aligned}
&\min \ r \\
&\text{s.t. } r \geq \sum_{u \in \mathcal{B}} d_{au}, \forall a \in I, \\
&\qquad d_{au} \geq \|a - u - x\| - M_{au}(1 - y_u), \forall a \in I, u \in \mathcal{B}, \\
&\qquad \sum_{u \in \mathcal{B}} y_u = p, \\
&\qquad r, d_{au} \geq 0, y_u \in \{0, 1\}.
\end{aligned}$$

Elzinga-Hearn Approach

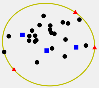


$LB = 3.4230$, $UB = 10.2715$
$Forbid = \{1, 17, 18\}$, $I = \{1, 2, 8\}$

Elzinga-Hearn Approach

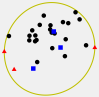


$LB = 3.4230$, $UB = 10.2715$
$Forbid = \{1, 17, 18\}$, $I = \{1, 2, 8\}$

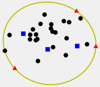


$LB = 10.2637$, $UB = 10.2909$
$Forbid = \{6, 12, 15\}$, $I = \{2, 8, 15\}$

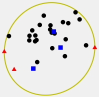
Elzinga-Hearn Approach



$LB = 3.4230$, $UB = 10.2715$
$Forbid = \{1, 17, 18\}$, $I = \{1, 2, 8\}$

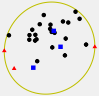


$LB = 10.2637$, $UB = 10.2909$
$Forbid = \{6, 12, 15\}$, $I = \{2, 8, 15\}$

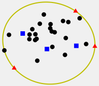


$LB = 10.2704$, $UB = 10.2705$
$Forbid = \{12, 14, 15\}$, $I = \{2, 8, 15\}$

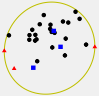
Elzinga-Hearn Approach



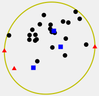
$LB = 3.4230$, $UB = 10.2715$
$Forbid = \{1, 17, 18\}$, $I = \{1, 2, 8\}$



$LB = 10.2637$, $UB = 10.2909$
$Forbid = \{6, 12, 15\}$, $I = \{2, 8, 15\}$

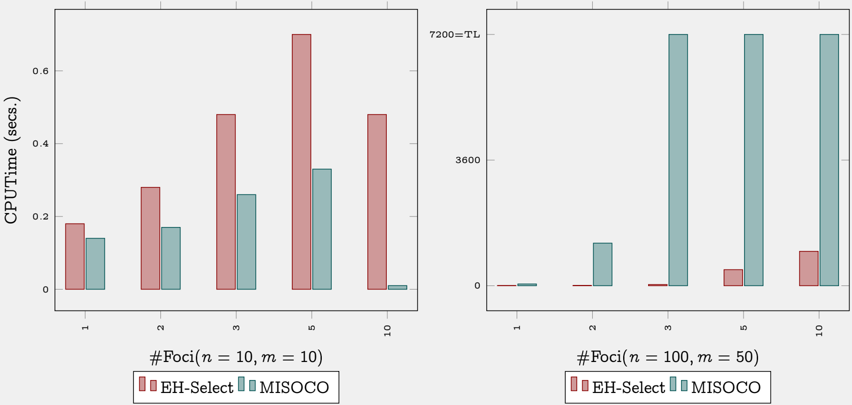


$LB = 10.2704$, $UB = 10.2705$
$Forbid = \{12, 14, 15\}$, $I = \{2, 8, 15\}$



$LB = 10.2715!!$, $UB = 10.2705$

Experiments: MISOCO vs. EH-Select



Max # Iterations: **3**

Conclusions

- We analyze polyellipses and their incorporation into classical location problem.
- We extend the Elzinga-Hearn algorithm to our problem.
- We also study the problem of selecting foci and adapt EH to it.
- Computational experiments show the efficiency of the approaches.

In progress...

- Polyellipses defined by polyhedral or ℓ_p-norms
- One-dimensional problem and Polyellipsoids.
- Complexity results.

Minimum Enclosing Polyellipses

Víctor Blanco
Universidad de Granada

(joint work with J. Puerto)